MATH 25B – PROBLEM SET #6
DUE TUESDAY MARCH 22ND

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Yan) separately from the problems from section 2 (which will be graded by Toly).

Standing assumptions

For this problem set, unless otherwise stated, you should work over the field $\mathbf{C}$ and assume that all vector spaces are finite-dimensional.

1. Yan's problems

(1) *Computing things*

(a) Compute the Jordan canonical form of

$$\begin{pmatrix} -3 & 2 & 0 & -6 \\ 0 & 3 & 0 & 0 \\ 2 & 1 & 3 & 2 \\ 6 & -3 & 0 & 9 \end{pmatrix}$$

(b) Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Find a matrix B in Jordan canonical form and an invertible matrix Q such that $B = QAQ^{-1}$.

(c) Are the matrices

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

similar?

(2) Let $T : V \to V$ be a linear transformation. Suppose that λ is an eigenvalue of T and that p is a polynomial.

(a) Is $p(\lambda)$ an eigenvalue of $p(T)$?

(b) Are all eigenvalues of $p(T)$ of this form?

(3) Let A and B be $n \times n$ matrices. Show that every non-zero eigenvalue of AB is an eigenvalue of BA, and conversely.

(4) *The set of invertible matrices is open (twice)*

(a) Show that the determinant of an $n \times n$ matrix gives a continuous function from $\mathbf{R}^{n^2}$ to $\mathbf{R}$. Deduce that the set of invertible matrices is open in $\mathbf{R}^{n^2}$.

(b) State and prove a formula for the inverse of the matrix

$$I - A$$

which is valid when every entry of the matrix A is sufficiently small. Use this to give another proof that the set of invertible matrices is open in $\mathbf{R}^{n^2}$.

Do the 1×1 *matrix case first.*

In fact the set of invertible matrices is also dense in the set of all matrices.

2. Toly's problems

(1) *Matrix differential equations*

(a) Show that if A is an $n \times n$ matrix, $x_1, \dots x_n$ are constant scalars, and $\dot{y}$ denotes the derivative of y with respect to t then

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \exp(At) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

is a solution to the matrix differential equation

$$(\star) \qquad \begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_n \end{pmatrix} = A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

(b) Suppose that

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

is any solution to $(\star)$. Compute the derivative of

$$\exp(-At) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

with respect to t. Deduce that there exist constants $x_1, \dots, x_n$ such that

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \exp(At) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

So the general solution of the system of linear differential equations $(\star)$ *is as in (a). We can compute this efficiently by diagonalising* A.

(2) *Higher-order linear differential equations and systems of linear equations*

(a) Suppose that the complex-valued function $f(t)$ satisfies the differential equation

$$f^{(n)} + a_{n-1} f^{(n-1)} + \ldots + a_1 f^{(1)} + a_0 f = 0$$

where $f^{(i)}$ denotes the ith derivative of f. Write down a matrix-valued differential equation satisfied by the vector

$$\begin{pmatrix} f \\ f^{(1)} \\ \vdots \\ f^{(n-1)} \end{pmatrix}.$$

(b) Use this to find all solutions to the differential equation

$$\frac{d^2 f}{dt^2} = -f.$$

(c) Show that the general *real-valued* solution to this differential equation is

$$f(t) = A\cos(t) + B\sin(t)$$

for some constants $A, B \in \mathbf{R}$.

This gives an algorithmic way to solve any linear n-th order differential equation with constant coefficients.

(3) *Jordan blocks and resonance*

Use the method above to find all real-valued solutions to the differential equations

$$y'' - 2y' - 3y = 0$$

and

$$y'' - 2y' + y = 0.$$

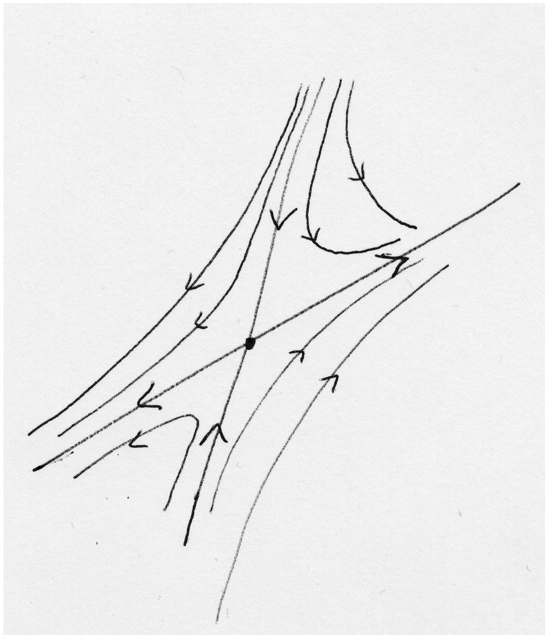
(4) *Autonomous dynamical systems*

In this problem we work in $\mathbf{R}^2$, and so work over the field $\mathbf{R}$.

(a) Suppose that

$$(\dagger) \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

where A is a 2×2 matrix with real eigenvalues, one strictly positive and one strictly negative. Show that solution trajectories $t \mapsto (x(t), y(t))$ near the origin look like



What are the two straight lines on this picture?
This picture is called the "phase portrait" of the autonomous dynamical system (†)*. It is called "autonomous" because the coefficients of* (†) *— the entries of A — do not vary with time.*

(b) Suppose that we change A. What are the other possible phase portraits?
There are lots of possibilities. A could have both eigenvalues positive, or two complex eigenvalues, or ...

(5) *A simple model of an epidemic*
A simple model of an epidemic in a city is as follows. Susceptible people enter the city at a constant rate of β per day. Infected people recover or die after a certain number of days. If they recover, they are immune. The number $S(t)$ of susceptible people at time t and the number $I(t)$ of infected people at time t satisfy the differential equations

$$\begin{aligned}\frac{dS}{dt} &= -\alpha SI + \beta \\ \frac{dI}{dt} &= -\sigma I + \alpha SI.\end{aligned}$$

Here α is a constant determined by the probability of an interaction between a susceptible person and an infected person, and $\frac{1}{\sigma}$ is the mean length of time that a person is infectious.

(a) Find all equilibrium points of the system — *i.e.* determine all constants (S_0, I_0) such that $S(t) = S_0$, $I(t) = I_0$ is a solution.

(b) Is the equilibrium solution stable? In other words, if you perturb the equilibrium solution a bit do you get a solution which tends back towards equilibrium.

In other other words, do solutions which start near the equilibrium solution converge to equilibrium or not? Does this depend on α, β, and σ?
This autonomous system is non-linear, but if you write $S = S_0 + \epsilon S(t)$, $I = I_0 + \epsilon I(t)$ *and take* ϵ *to be small then you can approximate it by a linear system.*

The second and third questions were suggested by Yan. I learned the last question from Dr Tom Judson.