MATH 25B – PROBLEM SET #7
DUE FRIDAY MARCH 25TH

1. Three problems

(1) *Square roots*
Does every matrix have a square root? In other words, if X is an $n \times n$ matrix, must there equal A such that $X = A^2$? And if $A^2 = B^2$, must $A = \pm B$?

(2) *Orthogonal and unitary matrices*
(a) Show that a real $n \times n$ matrix A has orthonormal columns if and only if $A^T A = I$.
(b) Show that a complex $n \times n$ matrix A has orthonormal columns if and only if $A^\dagger A = I$.
(c) Show that an $n \times n$ matrix has orthonormal rows if and only if it has orthonormal columns.

(3) *Orthonormal bases*
(a) Design an algorithm which takes as input a linearly independent subset $\{v_1, \ldots, v_k\}$ of an inner product space V and produces as output an orthogonal subset $\{w_1, \ldots, w_k\}$ such that
$$\operatorname{span}\{w_1, \ldots, w_r\} = \operatorname{span}\{v_1, \ldots, v_r\} \qquad \text{for } r = 1, 2, \ldots, k.$$
Figure out what to do for $k = 2$ first, then for $k = 3$.
(b) Prove that your algorithm works.
(c) Show that every finite-dimensional inner product space has an orthonormal basis.
The same sort of argument can be used to show that every orthonormal set in a finite-dimensional inner product space can be extended to an orthonormal basis.

2. Other problems

The problems in this section are purely optional — and in fact they will appear again as parts of assignments due later this semester. They are included in case you want extra practice problems for over break.

(1) *Jordan canonical form, but less mangled this time*
For this question, we work over an algebraically closed field. V is a finite-dimensional vector space and $T : V \to V$ is a linear transformation.
(a) Read the section on Jordan canonical form in Halmos.
You do not need to write anything for this bit.
(b) Show that if $\operatorname{im}(T) = \operatorname{im}(T^2)$ then $V = \operatorname{im}(T) \oplus \ker(T)$.

(c) Show that there exists a $k > 0$ such that $V = \text{im}(T^k) \oplus \ker(T^k)$.
So for each eigenvalue λ of T there exists a $k_\lambda > 0$ such that

$$V = \text{im}((T - \lambda I)^{k_\lambda}) \oplus \ker((T - \lambda I)^{k_\lambda}).$$

(d) Show that $\ker((T - \lambda I)^{k_\lambda})$ and $\text{im}((T - \lambda I)^{k_\lambda})$ are both T-invariant.
We are looking for a basis such that the matrix of T consists of Jordan blocks. So we want each basis element v to satisfy $(T - \lambda I)^r v = 0$ for some eigenvalue λ and some r.

(e) Suppose that $(T - \lambda I)^r(v) = 0$ for some $r > 0$. Show that $v \in \ker((T - \lambda I)^{k_\lambda})$.
So if we are looking for basis vectors which will make Jordan blocks with eigenvalue λ then we only need to look inside $\ker((T - \lambda I)^{k_\lambda})$. *For the rest of the problem, therefore, assume that $V = \ker((T - \lambda I)^{k_\lambda})$. Set $S = T - \lambda I$, so that S is nilpotent of index k_λ. We want to find a basis for V so that the matrix for S with respect to this basis has the form described in the second paragraph of page 112 of Halmos.*

(f) Consider the following algorithm. Start by setting
- $\beta = \emptyset$;
- $k = k_\lambda$.

Now perform the following steps;

(i) Pick vectors $y_1, \ldots, y_r$ such that

$$\{x + \ker(S^{k-1}) : x \in \beta\} \cup \{y_1 + \ker(S^{k-1}), \ldots, y_r + \ker(S^{k-1})\}$$

is a basis for $ker(S^{k_\lambda})/ker(S^{k-1})$. Set

$$\beta = \beta \cup \{S^{k-1}(y_1), S^{k-2}(y_1), \ldots, S(y_1), y_1\} \cup \ldots \cup \{S^{k-1}(y_r), S^{k-2}(y_r), \ldots, S(y_r), y_r\}.$$

In other words, we make sure β contains a basis for the space of vectors which die after k applications of S but not after fewer than k applications.

(ii) If β spans V, stop. Otherwise, decrease k until $\{x + \ker(S^{k-1}) : x \in \beta\}$ does not span $ker(S^{k_\lambda})/ker(S^{k-1})$.

(iii) Go to step (i).

Prove that this algorithm works. You need to show that
- at the end of step (ii), β is always LI;
- in step (ii), if we do not stop then k definitely decreases;
- the algorithm stops.

(2) Compute

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \vdots & & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{pmatrix}$$

This is known as a Vandermonde determinant.

(3) *QR factorization*
Show that every $n \times n$ real matrix A can be expressed as $A = QR$, where Q is orthogonal and R is upper-triangular. Is this factorization unique?