MATH 25B – PROBLEM SET #8
DUE TUESDAY 12TH APRIL

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Yan) separately from the problems from section 2 (which will be graded by Toly).

1. Yan's Problems

(1) *A vital lemma*
   (a) Problem 1-10 on page 5 of Spivak.
   (b) Show that any linear map $T : \mathbf{R}^n \to \mathbf{R}^m$ is continuous.
   There are linear maps between infinite-dimensional inner product spaces which are not continuous.

(2) *Differentiability implies continuity*
   Problem 2-1 on page 17 of Spivak.

(3) *Computing derivatives*
   (a) Problem 2-10 parts (b) and (d) on page 23 of Spivak.
   (b) Problem 2-11 part (b) on page 23 of Spivak.

(4) *Differentiating bilinear functions*
   Problem 2-12 on page 23 of Spivak.

(5) *QR factorization*
   Show that every $n \times n$ real matrix A can be expressed as $A = QR$, where Q is orthogonal and R is upper-triangular. Is this factorization unique?
   Think about the Gram–Schmidt process.

2. Toly's problems

(1) *Differentiating inner products*
   Problem 2-13 on page 23 of Spivak.

(2) *Computing partial derivatives*
   Problem 2-17 parts (b), (g), and (h) on page 28 of Spivak.

(3) *Partial derivatives of differentiable functions need not be continuous*
   Problem 2-32 on pages 33–34 of Spivak.
   I think we might have done part (a) last semester. If so, feel free to skip it.

(4) *The existence of partial derivatives does not imply differentiability*
Consider the function

$$\begin{aligned} f : \mathbf{R}^2 &\longrightarrow \mathbf{R} \\ (x, y) &\longmapsto \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \end{aligned}$$

(a) Show that the partial derivatives of f with respect to x and y exist everywhere on $\mathbf{R}^2$.
(b) At what points $(x, y) \in \mathbf{R}^2$ is f differentiable?

You should figure out why your answer to (b) does not contradict Theorem 2.8 on page 31 of Spivak.

(5) *The Five Lemma*
Suppose that each row of the diagram

$$\begin{array}{ccccccccc} V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & V_4 & \longrightarrow & V_5 \\ \downarrow{\scriptstyle T_1} & & \downarrow{\scriptstyle T_2} & & \downarrow{\scriptstyle T_3} & & \downarrow{\scriptstyle T_4} & & \downarrow{\scriptstyle T_5} \\ W_1 & \longrightarrow & W_2 & \longrightarrow & W_3 & \longrightarrow & W_4 & \longrightarrow & W_5 \end{array}$$

is an exact sequence of linear maps between vector spaces, and that $T_1, \ldots, T_5$ are linear maps such that T_1, T_2, T_4, and T_5 are isomorphisms. Show that T_3 is also an isomorphism.

This is surprisingly useful.