MATH 25B – PROBLEM SET #9
DUE TUESDAY 19TH APRIL

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Toly) separately from the problems from section 2 (which will be graded by Yan).

1. Toly's Problems

(1) *Maximization and minimization*
Let $f(x, y) = (3 - x)(3 - y)(x + y - 3)$.
(a) Make a sketch indicating the points (x, y) at which $f(x, y) \geq 0$.
(b) Find all points (x, y) such that $D_1 f(x, y) = D_2 f(x, y) = 0$.
Hint: $D_1 f(x, y)$ is divisible by $3 - y$.
(c) Which of the stationary points are local maxima? Which are local minima? Which are neither? Give reasons for your answers.
(d) Does f have an absolute minimum or absolute maximum on the whole of $\mathbf{R}^2$? Give reasons for your answers.

(2) *Directional derivatives and grad*
Let $f : \mathbf{R}^n \to \mathbf{R}$ be a differentiable function. For $a \in \mathbf{R}^n$, set

$$\nabla f(a) = \begin{pmatrix} D_1 f(a) \\ \vdots \\ D_n f(a) \end{pmatrix},$$

The vector $\nabla f(a)$ is called *the gradient of f at a* or *grad f at a*; the function

$$\nabla f : \mathbf{R}^n \to \mathbf{R}^n$$

is often called "grad f".
(a) Look at the definition of directional derivative $D_x f$ in problem 2-29 on page 33 of Spivak. Show that

$$D_x f(a) = (\nabla f(a), x)\,.$$

The parentheses here denote inner product; if you prefer we could write this as $D_x f(a) = \nabla f(a) \cdot x$.
(b) Show that $\nabla f(a)$ vanishes whenever a is a local maximum or local minimum of f.
(c) Show that $\nabla f(a)$ points in the direction of steepest increase of f at a. What does the length $\|\nabla f(a)\|$ represent?

(3) *Level surfaces*

(a) Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a smooth function. Suppose that

$$f(a, b, c) = k,$$

and that

$$\frac{\partial f}{\partial z}(a, b, c) \neq 0.$$

Show that there is an open set $U \subset \mathbf{R}^2$ containing (a, b) and a differentiable function $g : U \to \mathbf{R}$ such that

$$f(a, b, g(a, b)) = k.$$

Explain why the points in the set

$$\{(x, y, z) \in \mathbf{R}^3 : f(x, y, z) = k\}$$

which are sufficiently close to (a, b, c) form a smooth surface in $\mathbf{R}^3$.

(b) Suppose that $f : \mathbf{R}^3 \to \mathbf{R}$ is a smooth function, and that $k \in \mathbf{R}$ is such that the partial derivatives

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z}$$

do not simultaneously vanish anywhere on the set

$$f^{-1}(k) = \{(x, y, z) \in \mathbf{R}^3 : f(x, y, z) = k\}.$$

Explain why the set $f^{-1}(k)$ forms a smooth surface in $\mathbf{R}^3$. $f^{-1}(k)$ is called the level surface of f at level k.

You need to show that near each point $(a, b, c) \in f^{-1}(k)$, the set $f^{-1}(k)$ looks like a smooth surface in $\mathbf{R}^3$.

(c) Show that if $(a, b, c) \in f^{-1}(k)$ then $\nabla f(a, b, c)$ is perpendicular[1] to the surface $f^{-1}(k)$ at the point (a, b, c).

So "grad f points perpendicularly out of level surfaces of f".

(d) Draw a picture containing some level surfaces $f^{-1}(k)$ and some values of ∇f for

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Can you see how your answers to 3(c) and 4(c) fit the picture?

It might help to do part (d) of this question first.

[1]We say that a vector $v \in \mathbf{R}^3$ is perpendicular to a surface $S \subset \mathbf{R}^3$ at the point x in S if and only if for each smooth curve

$$\gamma : \mathbf{R} \longrightarrow \mathbf{R}^3$$
$$t \longmapsto \gamma(t)$$

such that $\gamma(t) \in S$ for all t and $\gamma(0) = x$ the vector $\gamma'(0)$ is perpendicular to v. You may want to draw a few pictures to work out why this is the right definition of "perpendicular to a surface" when the surface is not a plane.

2. Yan's Problems

(1) *Visualizing surfaces*
Sketch level surfaces of
$$f(x, y, z) = x^2 - y^2 + 2z^2$$
at levels -2, -1, 0, 1, and 2.
One way to approach this is to sketch the intersections of the surface with various co-ordinate planes $x = const$, $y = const$, or $z = const$.

(2) *Lagrange Multipliers*
Let f and g be smooth functions from $\mathbf{R}^3$ to $\mathbf{R}$. In this question we will develop a method for solving the following *constrained maximization problem*: find the maximum value $f(x, y, z)$ where we only consider points (x, y, z) such that $g(x, y, z) = k$.
We will assume throughout that:
- ∇g is not zero anywhere on $g^{-1}(k)$, so in particular the level set $g^{-1}(k)$ is a smooth surface;
- all level sets $f^{-1}(l)$ are smooth surfaces[2].

Let M be the maximum value of f on the surface $g^{-1}(k)$ — this is what we want to find. Suppose that this maximum value occurs at the point $(a, b, c) \in g^{-1}(k)$.
(a) Show that the level surfaces $f^{-1}(M)$ and $g^{-1}(k)$ are tangent at (a, b, c).
(b) Show that the point (a, b, c) solves the system of equations
$$(\star) \qquad \begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) & \text{for some } \lambda \in \mathbf{R} \\ g(x, y, z) = k \end{cases}$$
(c) Must any solution to $(\star)$ give either the maximum or minimum value of $f(x, y, z)$ on the set $g^{-1}(k)$?
(d) By solving the system $(\star)$ with appropriate choices of f and g, find the minimum surface area of a cuboid box which has volume equal to 8 cubic units. Is there such a box with maximum surface area?

Depending on how you think about this, you may want to solve (b) before you solve (a). This technique is called the method of Lagrange multipliers: λ is called a Lagrange multiplier.

(3) *Maximization and minimization on a region with boundary*
Find the maximum and minimum values of
$$f(x, y) = x^3 - 3xy^2 + y^3$$
on the region $x^2 + y^2 \leq 1$.
Either the extrema are in the interior or on the boundary. If they are on the boundary, you can find them either using Lagrange multipliers or by parametrizing the boundary.

[2]We can relax this to "all level sets of f which meet a neighbourhood of the surface $g^{-1}(k)$ are smooth surfaces inside that neighbourhood of $g^{-1}(k)$".

(4) *Injectivity and local injectivity*
Problem 2-38 on pages 39–40 of Spivak. Show in addition that for the function f in part (b), we can find a neighbourhood U of any point $(a, b) \in \mathbf{R}^2$ such that $f|_U$ is injective. So f is *locally injective* but not injective.

3. Something to think about

(1) This problem is optional and will not be graded.
How and under what conditions can we generalize the method of Lagrange multipliers to solve the problem: find the maximum value of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = l$?

The first of Toly's problems is taken from volume II of *Calculus* by Tom Apostol. This is an excellent book and I urge you to take a look at it.