# MATH 25B – PROBLEM SET #10
# DUE TUESDAY 26TH APRIL

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Yan) separately from the problems from section 2 (which will be graded by Toly).

## 1. Yan's Problems

(1) *When are functions integrable?*
The proof of Theorem 3–8 presented in Spivak is flawed. Expand the brief sketch on page 145 into a correct proof of the following statement: a bounded function $f : A \to \mathbf{R}$, where $A \subset \mathbf{R}^n$ is a closed rectangle, is integrable if and only if the set

$$\{a \in A : f \text{ is not continuous at } a\}$$

has measure zero.

(2) *Measure zero*
Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Show that the graph of $f$, which is a subset of $\mathbf{R}^2$, has measure zero.
*The analogous statement about the graphs of continuous functions $f : \mathbf{R}^n \to \mathbf{R}^m$ is also true, and for much the same reason.*

(3) *Content zero subsets do not affect integrals*
Suppose that $A$ is a subset of $\mathbf{R}^n$, that $B$ is a subset of $A$ such that $A - B$ has content zero, that $f : A \to \mathbf{R}$ is a bounded function, and that $g : B \to \mathbf{R}$ is the restriction of $f$ to $B$. Show that

$$\int_A f \text{ exists if and only if } \int_B g \text{ exists}$$

and that if the integrals exist then they are equal.

(4) *Some integrals*
(a) Prove that if $f(x, y)$ is a continuous function and $R$ is the region

$$R = \{(x, y) \in \mathbf{R}^2 : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where $g_1(x) \leq g_2(x)$ for all $x \in [a, b]$ and the functions $g_1$, $g_2$ are continuous, then

$$\int_R f(x, y)\, dx\, dy = \int_{x=a}^{x=b} \left( \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy \right) dx.$$

We could call such regions "swept out by vertical lines" State a similar result for regions swept out by horizontal lines.

(b) Sketch the region $S = \{(x, y) \in \mathbf{R}^2 : |x| + |y| \leq 1\}$ and evaluate

$$\int_S e^{x+y}\, dx\, dy.$$

(c) Sketch the solid bounded by the surface $z = x^2 - y^2$, the $xy$-plane, and the planes $x = 1$ and $x = 3$, and compute its volume.

## 2. Toly's Problems

(1) *Switching the order of integration*

(a) Compute

$$\int_{x=0}^{x=1} \int_{y=x}^{y=1} e^{y^2}\, dy\, dx$$

by writing it as an iterated integral in the other order: integrating $x$ first and then $y$.

*Note that this is certainly not equal to*

$$\int_{y=x}^{y=1} \int_{x=0}^{x=1} e^{y^2}\, dy\, dx.$$

(b) When computing the volume of a solid region under the paraboloid $z = x^2 + y^2$ and above a region $S$ of the $xy$-plane, the following sum of iterated integrals was obtained

$$V = \int_{y=0}^{y=1} \left( \int_{x=0}^{x=y} (x^2 + y^2)\, dx \right) dy + \int_{y=1}^{y=2} \left( \int_{x=0}^{x=2-y} (x^2 + y^2)\, dx \right) dy.$$

Sketch the region $S$ and express $V$ as an iterated integral in which the order of integration is reversed. Compute $V$.

(2) *A famous integral and an infamous integral*

(a) Parts (d) and (e) of Problem 3–41 on pages 73–74 of Spivak.

(b) A satellite has a smooth unbroken skin made up of portions of two circular cylinders of equal diameters $D$ whose axes meet at right angles. It is proposed to transport the satellite in a cubical packing box of inner dimension $D$. Show that one-third of the box will be wasted space.

*Note that, in view of the quote on page 74, I am definitely not a mathematician.*

(3) *Cylindrical and spherical polar co-ordinates*

(a) Find out what the cylindrical and spherical polar co-ordinate systems on $\mathbf{R}^3$ are. Use the change of variable formula to derive analogs of the relation

$$\text{“}dx\, dy = r\, dr\, d\theta\text{”}$$

for cylindrical and spherical polar co-ordinates.

(b) Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 5$ and below by the paraboloid $x^2 + y^2 = 4z$.

(c) Determine the mass of the solid lying between two concentric hemispheres of radii $a$ and $b$, where $0 < a < b$, if the density at each point is equal to the square of the distance of that point from the center.

(4) *Clairault's Theorem*
Problem 3–28 on page 61 of Spivak.

(5) *Change of variable*
Evaluate
$$\int_T \exp\left(\frac{x+y}{x-y}\right)\, dx\, dy$$
where $T \subset \mathbf{R}^2$ is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,-1)$, and $(0,-2)$.

Questions 4(b) and 4(c) in Yan's section, and 1(b), 2(b), 3(b), and 3(c) in Toly's section are based on material from volume 2 of *Calculus* by Tom Apostol. Question 5 on Yan's section is based on material from *Multivariable Calculus* by Stewart.