MATH 25B – PROBLEM SET #11
DUE TUESDAY 3RD MAY

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Toly) separately from the problems from section 2 (which will be graded by Yan).

1. TOLY'S PROBLEMS

(1) *The wedge product is supercommutative*
   Let V be an n-dimensional vector space, let $\omega \in \Omega^k(V)$, and let $\eta \in \Omega^l(V)$.
   (a) Show that

$$\eta \wedge \omega(v_1, \ldots, v_{k+l}) = \frac{1}{k!\, l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \omega(v_{\tilde{\sigma}(1)}, \ldots, v_{\tilde{\sigma}(k)}) \eta(v_{\tilde{\sigma}(k+1)}, \ldots, v_{\tilde{\sigma}(k+l)})$$

where

$$\tilde{\sigma}(1) = \sigma(l+1) \qquad\qquad \tilde{\sigma}(k+1) = \sigma(1)$$
$$\tilde{\sigma}(2) = \sigma(l+2) \qquad\qquad \tilde{\sigma}(k+2) = \sigma(2)$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$\tilde{\sigma}(k) = \sigma(l+k) \qquad\qquad \tilde{\sigma}(k+l) = \sigma(l)$$

   So we need to relate $\operatorname{sgn}(\sigma)$ and $\operatorname{sgn}\tilde{\sigma}$.
   (b) Find a permutation π such that $\tilde{\sigma} = \sigma\pi$.
   (c) Deduce a relationship between $\operatorname{sgn}(\sigma)$ and $\operatorname{sgn}\tilde{\sigma}$, and show that

$$\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta.$$

(2) *Wedge products and determinants*
   (a) Problem 4–1 on page 84 of Spivak.
   (b) What is the relationship between $\varphi_{i_1} \wedge \ldots \varphi_{i_k}$ and $\operatorname{Alt}(\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_k})$?
      Just state the relationship here — you do not need to include a proof.
   This establishes an assertion made as part of the proof that

$$\{\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_k} : 1 \le i_1 < i_2 < \ldots < i_k \le n\}$$

   is a basis for $\Omega^k(\mathbf{R}^n)$.

(3) *Vector fields and flowlines*
   A curve

$$\gamma : \mathbf{R} \longrightarrow \mathbf{R}^n$$
$$t \longmapsto \gamma(t)$$

   is called a *flowline* of a vector field[1] $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ if and only if

$$\gamma'(t)(1) = \mathbf{F}(\gamma(t)) \qquad \text{for all } t \in \mathbf{R}.$$

   Note that if $\gamma(t)$ represents the position of a particle at time t then the left-hand side here is the velocity of the particle at time t and the right-hand side is the value of the vector field $\mathbf{F}$ at the point $\gamma(t)$, so that if $\mathbf{F}$ is the velocity field of a fluid then a flowline γ represents the path of a particle moving in the fluid.
   (a) Sketch the vector field on $\mathbf{R}^2$ given by

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ xy \end{pmatrix}$$

   and sketch the flowlines through $(1,0)$, $(-4,0)$, and $(2,2)$.
   (b) For each of theses flowlines, compute the function $\gamma(t)$.
   (c) Sketch the vector field ∇f, where

$$f : \mathbf{R}^2 \longrightarrow \mathbf{R}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto x^2 + 4y^2$$

   together with some flowlines for ∇f.
   These flowlines are the paths along which f increases most rapidly.

(4) *Curl*
   Consider the solid cylinder in $\mathbf{R}^3$ given by

$$S = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq 1\}$$

   and the vector field $\mathbf{v}$ on S given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

   (a) Sketch the vector field $\mathbf{v}$.
      $\times$ *is the cross product of vectors in $\mathbf{R}^3$, defined for instance at*

      http://planetmath.org/encyclopedia/CrossProduct.html.

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[1]We are abusing notation here: a vector field on $\mathbf{R}^n$ is a section of the tangent bundle $T\mathbf{R}^n$ of $\mathbf{R}^n$, so we really mean

$$\mathbf{F} : \mathbf{R}^n \longrightarrow T\mathbf{R}^n$$
$$p \longmapsto (p, \mathbf{F}(p)).$$

In other words, we regard the vector $\mathbf{F}(p)$ as an element of the tangent space $T_p\mathbf{R}^n$.

$\mathbf{v}$ *is the velocity field produced by rotating the (rigid) cylinder S about the z-axis with constant angular speed ω.*

(b) Compute curl $\mathbf{v}$.

Vector fields which curl *to zero are called irrotational.*

(c) Compute curl $\mathbf{F}$, where

$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6xz \\ 2y \\ 3x^2 \end{pmatrix}$$

So if $\mathbf{F}$ is the velocity vector field of a fluid then curl $\mathbf{F}$ *at a point $p \in \mathbf{R}^3$ points along the axis about which the fluid is rotating (locally near p) and has length proportional to the speed of rotation at p.*

(5) *Divergence*

(a) Compute the divergence of the vector field from 4(c) above.

(b) Compute the divergence of ∇f, where $f : \mathbf{R}^3 \to \mathbf{R}$ is smooth.

This Laplacian *came up when we were talking about hearing the shape of a drum.*

(c) Let $\mathbf{F}$ be the velocity vector field of a fluid in $\mathbf{R}^3$. Explain the relationship between div $\mathbf{F}$ at $p \in \mathbf{R}^3$ and the limit as $\epsilon \to 0$ of the net flow of fluid out of a cube of side 2ϵ centered at p.

Vector fields $\mathbf{F}$ such that div $\mathbf{F}$ *is zero are called incompressible.*

2. Yan's Problems

(1) *Product rule*

Let $f, g : \mathbf{R}^n \to \mathbf{R}$ be smooth functions. Show that

$$d(f \cdot g) = f\, dg + g\, df.$$

(2) *Motion on a sphere*

Problem 4–16 on page 96 of Spivak.

(3) *d generalizes* div, grad, *and* curl

(a) Show that the following diagram is commutative:

$$0 \longrightarrow \text{0-forms} \xrightarrow{d} \text{1-forms} \xrightarrow{d} \text{2-forms} \xrightarrow{d} \text{3-forms} \longrightarrow 0$$

with vertical maps R, S, T, U going up, and

$$0 \longrightarrow \text{functions} \xrightarrow{\text{grad}} \text{vector fields} \xrightarrow{\text{curl}} \text{vector fields} \xrightarrow{\text{div}} \text{functions} \longrightarrow 0$$

where

• all vector fields, differential forms, and functions are on $\mathbf{R}^3$;

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$$R(f) = f$$

$$S \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix} = F_1\,dx + F_2\,dy + F_3\,dz$$

$$T \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix} = F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$$

$$U(f) = f\,dx \wedge dy \wedge dz$$

(b) Deduce that

$$\mathrm{curl}(\mathrm{grad}(f)) = 0$$

for all functions $f : \mathbf{R}^3 \to \mathbf{R}$, and that

$$\mathrm{div}(\mathrm{curl}(\mathbf{F})) = 0$$

for all vector fields $\mathbf{F}$ on $\mathbf{R}^3$.

(c) A vector field $\mathbf{F}$ is called *conservative* if and only if $\mathbf{F} = \mathrm{grad}(f)$ for some function f. Is the vector field

$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 y \\ 2y \\ 3x^2 \end{pmatrix}$$

conservative?

If it exists, f is called the potential function *for* $\mathbf{F}$.

We can write down a similar diagram for vector fields, differential forms, etc. defined on any sufficiently nice subset $A \subset \mathbf{R}^3$. The rows in such a diagram are in general not exact, but as $d^2 = 0$ we know that $\mathrm{im}\,d \subset \ker d$ at each stage. The quotient space

$$\ker d / \operatorname{im} d \qquad \text{where } \ker d, \operatorname{im} d \subset \{i\text{-forms on } A\}$$

is known as the ith deRham cohomology group of A and written $H^i(A; \mathbf{R})$. Remarkably, these quotient spaces are finite-dimensional.

(4) *Cohomology groups feel the shape of the domain*
  Consider the 1-form

$$\omega = -\frac{y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy$$

  defined on $A = \mathbf{R}^2 - \{(0,0)\}$.
  (a) Is ω closed? In other words, does $d\omega = 0$?
  (b) Is ω exact? In other words, does there exist a function $f : A \to \mathbf{R}$ such that $df = \omega$?

(c) Is the restriction of ω to
$$B = \mathbf{R}^2 - \{(x, 0) : x \geq 0\}$$
exact?

You should, of course, prove that your assertions here are correct.