

MATH 25B – PROBLEM SET #12
DUE WEDNESDAY 11TH MAY

Half of this assignment will be graded by Yan and the other half will be graded by Toly. Please turn in the problems from section 1 (which will be graded by Yan) separately from the problems from section 2 (which will be graded by Toly).

1. YAN'S PROBLEMS

(1) *Gauss's Law*

The electric field \mathbf{E} at a point $\mathbf{x} \in \mathbf{R}^3$ not at the origin created by a point charge of charge Q at the origin is equal to

$$\frac{Q\epsilon\mathbf{x}}{\|\mathbf{x}\|^3}$$

where ϵ is a constant. (This is the electric force experienced by a unit charge — a “test charge” — placed at \mathbf{x} .)

- (a) Write down the 2-form corresponding to the vector field E . Is it closed?
- (b) Compute the flux of E outwards through a sphere of radius $r > 0$ centred at the origin.
You should do this directly.
- (c) Why do (a) and (b) not contradict Stokes's Theorem?
- (d) Explain why Stokes's Theorem implies *Gauss's Law*: the electric flux out of a smooth closed surface $S \subset \mathbf{R}^3$ is proportional to the charge enclosed by the surface.

(2) *Line Integrals*

- (a) Compute the work done by a force field

$$\mathbf{F}(x, y, z) = \begin{pmatrix} y^2 \cos z \\ 2xy \cos z \\ -xy^2 \sin z \end{pmatrix}$$

moving a particle from $(2, 0, 0)$ to $(0, 0, 3)$ along the curve which is the intersection of the half-ellipsoid

$$9x^2 + y^2 + 4z^2 = 36, \quad z \geq 0$$

with the xz -plane.

- (b) Suppose that \mathbf{F} is a velocity field of a fluid moving in \mathbf{R}^2 such that one of the flowlines of \mathbf{F} is a closed curve. Can \mathbf{F} be conservative?

(3) *Stokes's Theorem*

(a) Use Stokes's Theorem to evaluate

$$\int_C -y^2 dx + x dy + z^2 dz$$

where C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented anti-clockwise when viewed from above.

(b) Let C be a smooth closed curve in \mathbf{R}^2 , oriented anticlockwise. Show that

$$\int_C x dy = - \int_C y dx = \frac{1}{2} \int_C x dy - y dx = \text{the area bounded by } C.$$

What is the area of the polygon in \mathbf{R}^2 with vertices $(x_1, y_1), \dots, (x_n, y_n)$?

(4) *Eureka!*

Problem 5–36 on page 137.

“Three-manifold with boundary” in this case just means bounded region in \mathbf{R}^3 bounded by a smooth surface S .

(5) *If only the surface were closed...*

Let

$$\omega = z^2 x dy \wedge dz - \left(\frac{1}{3}y^3 + \tan z\right) dx \wedge dz + (x^2 z + y^2) dx \wedge dy.$$

Compute

$$\int_S \omega$$

where S is the top half of the sphere $x^2 + y^2 + z^2 = 1$ (including the boundary), oriented so that the boundary circle is oriented anti-clockwise when viewed from above.

... then we could use Stokes's Theorem.

2. TOLY'S PROBLEMS

(1) *By popular demand*

Let $f : \mathbf{R}^n \rightarrow \mathbf{R}$ be such that it and all its first and second partial derivatives are differentiable at each point of an open ball B centred at $x \in \mathbf{R}^n$.

(a) Show that, for $h = (h^1, \dots, h^n) \in \mathbf{R}^n$ sufficiently small there is some point y on the straight line between x and $x + h$ such that

$$f(x+h) = f(x) + \sum_{i=1}^{i=n} \frac{\partial f}{\partial x_i}(x) h^i + \frac{1}{2!} \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) h^i h^j + \frac{1}{3!} \sum_{i=1}^{i=n} \sum_{j=1}^{j=n} \sum_{k=1}^{k=n} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(y) h^i h^j h^k$$

One way to do this is to apply Taylor's Theorem to the function

$$\begin{aligned} g : \mathbf{R} &\longrightarrow \mathbf{R} \\ t &\longmapsto f(x + th) \end{aligned}$$

Note that

$$\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x) h^i h^j = h^T M h$$

where the matrix M has (i, j) entry $\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$.

(b) Suppose $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ is smooth and that $(x, y) \in \mathbf{R}^2$ is a critical point:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

Let M be the matrix defined above. Show that M is diagonalizable. Let

$$D = \det M$$

$$T = \text{tr } M$$

Show that

- if $D > 0$ and $T > 0$, x is a local minimum;
- if $D > 0$ and $T < 0$, x is a local maximum;
- if $D < 0$, x is a saddle point.

If $D = 0$ then we would need to look at the third derivatives to work out what is going on. Let's not.

(c) Find and classify the critical points of the function

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

Homology

A sequence of vector spaces and linear maps

$$\dots \xrightarrow{T_{i-1}} V_i \xrightarrow{T_i} V_{i+1} \xrightarrow{T_{i+1}} V_{i+2} \xrightarrow{T_{i+2}} \dots$$

is called a *complex* if $T_{i+1} \circ T_i = 0$ for all i . Exact sequences are examples of complexes. The sequence

$$0 \longrightarrow \Omega_{dR}^0(A) \xrightarrow{d} \Omega_{dR}^1(A) \xrightarrow{d} \Omega_{dR}^2(A) \xrightarrow{d} \Omega_{dR}^3(A) \longrightarrow 0$$

for an open subset A of \mathbf{R}^3 is another example, called the *de Rham complex* of A . Given a complex, we define the *i th homology group* of the complex to be the quotient vector space

$$H_i = \ker T_i / \text{im } T_{i-1}.$$

Note that this makes sense, as $\text{im } T_{i-1}$ is a subspace of $\ker T_i$. A complex is exact if and only if all its homology groups are zero.

The *i th homology group* of the de Rham complex of A is called the *i th de Rham cohomology group* of A , and written

$$H_{dR}^i(A).$$

Roughly speaking, it measures how many closed i -forms on A are not exact.

(2) *Zeroth cohomology*

Suppose that A is a connected open subset of \mathbf{R}^n . Compute $H_{dR}^0(A)$. What happens if A is not connected?

(3) *The Snake Lemma*

Suppose that we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & A_{k+1} & \xrightarrow{f_{k+1}} & B_{k+1} & \xrightarrow{g_{k+1}} & C_{k+1} & \longrightarrow & 0 & & \\
 & & \uparrow d_k^A & & \uparrow d_k^B & & \uparrow d_k^C & & & & \\
 0 & \longrightarrow & A_k & \xrightarrow{f_k} & B_k & \xrightarrow{g_k} & C_k & \longrightarrow & 0 & & \\
 & & \uparrow d_{k-1}^A & & \uparrow & & \uparrow d_{k-1}^C & & & & \\
 0 & \longrightarrow & A_{k-1} & \xrightarrow{f_{k-1}} & B_{k-1} & \xrightarrow{g_{k-1}} & C_{k-1} & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
 \end{array}$$

where the rows are exact and the columns are complexes. Such a thing is known as a *short exact sequence of complexes*.

(a) Show that the maps

$$\begin{array}{ccc}
 f_* : H_k(A) \longrightarrow H_k(B) & & g_* : H_k(B) \longrightarrow H_k(C) \\
 [\omega] \longmapsto [f_k(\omega)] & & [\rho] \longmapsto [g_k(\rho)]
 \end{array}$$

are well-defined. Here $\omega \in \ker(d_k^A)$ and $[\omega]$ is the coset $\omega + \text{im}(d_{k-1}^A)$ in the quotient space $H_k(A) = \ker(d_k^A) / \text{im}(d_{k-1}^A)$, and similarly for $[\rho]$.

(b) Construct a map $h_* : H_k(C) \longrightarrow H_{k+1}(A)$ such that the sequence

$$\dots \longrightarrow H_k(A) \longrightarrow H_k(B) \longrightarrow H_k(C) \longrightarrow H_{k+1}(A) \longrightarrow H_{k+1}(B) \longrightarrow \dots$$

is exact.

To do this, start with $\omega \in \ker(d_k^C)$ and try to construct an element $\nu \in \ker(d_{k+1}^A)$, well-defined up to an ambiguity of the form $\nu \rightsquigarrow \nu + d_k^A(\phi)$, and then check that if you change ω by an element of $\text{im}(d_{k-1}^C)$ then $[\nu] \in H_{k+1}(A)$ doesn't change.

In other words, from a short exact sequence of complexes one can construct a long exact sequence of homology groups.

(4) *The Mayer–Vietoris sequence*

Suppose that U and V are open subsets of \mathbf{R}^n , and that $M = U \cup V$. Any differential form on $U \cup V$ restricts to give differential forms on U and on V ; similarly, any differential form on U or on V restricts to give a form on $U \cap V$. So we have a diagram

$$0 \longrightarrow \Omega_{dR}^k(U \cup V) \longrightarrow \Omega_{dR}^k(U) \oplus \Omega_{dR}^k(V) \xrightarrow{T} \Omega_{dR}^k(U \cap V) \longrightarrow 0$$

where the map T sends a pair of forms (ω, η) to $\omega|_{U \cap V} - \eta|_{U \cap V}$. This sequence is clearly exact everywhere except at the last step: to show that the sequence is exact, we need to show that T is surjective.

- (a) Write $\mathbf{R} = U' \cup V'$, where $U' = (-1, \infty)$ and $V' = (\infty, 1)$. Show that any 0-form (i.e. smooth function) $f : U' \cap V' \rightarrow \mathbf{R}$ can be written as $g|_{U' \cap V'} - h|_{U' \cap V'}$ where $g : U' \rightarrow \mathbf{R}$ and $h : V' \rightarrow \mathbf{R}$ are smooth functions.

It may help to use a partition of unity: we know (from reading) that there exist smooth functions $\rho_{U'} : \mathbf{R} \rightarrow \mathbf{R}$ and $\rho_{V'} : \mathbf{R} \rightarrow \mathbf{R}$ such that $\rho_{U'} + \rho_{V'} = 1$, $\rho_{U'}$ is zero outside U' , and $\rho_{V'}$ is zero outside V' .

- (b) Prove that the sequence above is exact.

You may assume anything that you need about the existence of partitions of unity.

- (c) Show that there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 &\longrightarrow H_{dR}^0(U \cup V) \longrightarrow H_{dR}^0(U) \oplus H_{dR}^0(V) \longrightarrow H_{dR}^0(U \cap V) \longrightarrow \dots \\ \dots &\longrightarrow H_{dR}^1(U \cup V) \longrightarrow H_{dR}^1(U) \oplus H_{dR}^1(V) \longrightarrow H_{dR}^1(U \cap V) \longrightarrow \dots \\ \dots &\longrightarrow H_{dR}^2(U \cup V) \longrightarrow H_{dR}^2(U) \oplus H_{dR}^2(V) \longrightarrow H_{dR}^2(U \cap V) \longrightarrow \dots \end{aligned}$$

(5) *The de Rham cohomology of the punctured plane*

- (a) Let $A = \mathbf{R}^2 - \{(0, 0)\}$. By writing A as the union of star-shaped open sets, compute the de Rham cohomology groups $H_{dR}^i(A)$.

(b) Compute the de Rham cohomology groups of the plane with n points removed. *So there is no smooth bijection between $\mathbf{R}^2 - \{n \text{ points}\}$ and $\mathbf{R}^2 - \{m \text{ points}\}$ unless $n = m$.*

3. BEAUTIFUL THINGS

This problem is purely optional and will not affect your grade.

- (1) (a) Look at question 4–20 on page 96 of Spivak.

You do not need to write anything for this bit.

- (b) Check that your argument in the last question also computes the de Rham cohomology groups of the subset $B \times [a, b]$ of \mathbf{R}^3 , where

$$B = \{(x, y) : 1 \leq x^2 + y^2 \leq 2\}$$

is an annulus in the plane.

You do not need to write anything for this bit either.

- (c) Compute the de Rham cohomology groups of the spherical shell

$$S = \{(x, y, z) \in \mathbf{R}^3 : 1 - \epsilon \leq x^2 + y^2 + z^2 \leq 1\} \quad \epsilon > 0 \text{ is small}$$

Write S as the union of the top two-thirds of the S and the bottom two-thirds of S . Now there is a smooth bijection from the top two-thirds of S to a star-shaped domain. . .

- (d) Let S_h be a slightly-thickened version of a closed surface in \mathbf{R}^3 with h “holes” in it, so S_0 is the thickened sphere from (c), S_1 looks like the surface of a bagel, S_2 looks like

<http://mathworld.wolfram.com/DoubleTorus.html>

and so on. Use the Mayer–Vietoris sequence to compute the de Rham cohomology groups of S_h .

Because de Rham cohomology is what is called homotopy invariant — roughly speaking, it does not change under continuous deformation — the de Rham cohomology groups of S_h are the same as those of the unthickened version of S_h . So this result implies that there is no smooth bijection between the closed surface with n holes and the closed surface with m holes unless $n = m$.

Many of the questions in Yan’s section are based on material from *Calculus, Early Transcendentals* by Stewart. The Mayer–Vietoris question is modelled on the discussion in the excellent *Differential Forms in Algebraic Topology* by Bott and Tu.