Final Exam — Math 134

Elizabeth Denne

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1 Instructions

This is the take home final exam for Math 134. You are allowed to use any books or notes that you wish. You may also talk to me or email me about the exam. However you are **not** allowed to discuss this exam with any of your classmates.

This exam must be handed in to me by **5pm Friday January 14th 2005**.

Please make sure that your name is on the solutions. Please also staple your solutions - I don't want to lose any of your work! To make grading easier, please write your solutions in the order that they appear on the exam.

2 The Questions

1. (15=5+10 points) (a) What is the definition of a differentiable manifold?

Now please do either (b) or (c):

(b) Given M is an m-dimensional manifold and N is an n-dimensional manifold, prove that $M \times N$ is a differentiable $(n+m)$-dimensional manifold. Hint: a coordinate neighborhood is of the form $(U \times V, \phi \times \psi)$, where (U, ϕ) (V, ψ) are coordinate neighborhoods of M and N respectively and $(\phi \times \psi)(p, q) = (\phi(p), \psi(q))$ in $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$.
(c) Prove that the tangent bundle TM is a differentiable $2n$-dimensional manifold, where M is an n-dimensional manifold. Hint: Given a coordinate neighborhood (U, ϕ) on M, a coordinate neighborhood of TM is $(\tilde{U}, \tilde{\phi})$. Here $\tilde{U}$ is all vectors X_p, such that $p \in U$ and $\tilde{\phi}(X_p) = (\phi(p), y^1, \ldots, y^n)$, where $\phi(p) = (x^1, \ldots, x^n)$ are the coordinates of p, $X_p = \Sigma y^i \frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^i}$ are the coordinate frames.

HINT: It is OK to ask me about part (a). That way you can be sure to check all the right parts of the definition when you do the next part.

2. (15=4+7+4 points) Let M = all 2×2 matrices $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ with real coefficients and $\det A = 1$ and AA^T diagonal (that is, off-diagonal entries all 0).

(a) Describe M as the solution set of 2 equations in 4 variables.

(b) Show M is a manifold.

(c) Let $M = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : g^1(x^1, x^2, x^3, x^4) = 0,\ g^2(x^1, x^2, x^3, x^4) = 0\}$. At $x_0 = (x_0^1, x_0^2, x_0^3, x_0^4) \in M$, $\frac{\partial(g^1,g^2)}{\partial(x^1,x^2,x^3,x^4)}|_{x_0} = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 0 \end{pmatrix}$.

Around x_0 in M, ???? are smooth functions of ????. Fill in the ???? and give a short explanation of your answer.

3. (15= 3+4+4+4 points) Let M be an n-dimensional manifold.

(a) What does it mean to say M is orientable? (I'm looking for the first definition we gave in class. If you are uncertain what I mean, please check with me.)

(b) If α is an n-form on M, give the coordinate expression for α in a coordinate chart $(U, \phi = (x^1, x^2, x^3, \ldots, x^n))$. How is it related to the expression for α in the coordinate chart $(U, (x'^1, x'^2, \ldots, x'^n) = (x^2, x^1, x^3, \ldots, x^n))$?

(c) How is it related to the expression for α in $(y^1, y^2, \ldots, y^n)$ on $U \cap V$, if $(V, \psi = (y^1, y^2, \ldots, y^n))$ is another coordinate chart?

(d) If there is a **never-zero** n-form α on M, show that M is orientable (as in the definition in part (a)).
HINT: If you get stuck, try this for $n = 2$ or $n = 3$.

4. (30=3+10+4+3+4+5 points) On $M = \mathbb{R}^3 = \{(x^1, x^2, x^3)\}$, let $\alpha = x^1\, dx^2 \wedge dx^3 - x^2\, dx^3 \wedge dx^1 + x^3\, dx^1 \wedge dx^2$. Define $\gamma : [0,1]^3 = \{(u^1, u^2, u^3) : 0 \le u^i \le 1\} \to \mathbb{R}^3$ by:

$$x^1 = (4 + u^3 \cos(2\pi u^1)) \cos(2\pi u^2),$$

$$x^2 = (4 + u^3 \cos(2\pi u^1)) \sin(2\pi u^2),$$

$$x^3 = u^3 \sin(2\pi u^1).$$

(a) Find $d\alpha$.

(b) Write $\partial\gamma$ as a signed sum of 2-cubes. Give the formula for each 2-cube.

(c) Why can your answer in (b) be simplified to only 2 terms?

(d) Show $\gamma_{3,0}^*\, \alpha = 0$.

(e) Therefore $\int_{\partial\gamma} \alpha = \int_{\gamma_{3,1}} \alpha$. You have seen $\gamma_{3,1}$ before; sketch the surface it covers. (Please draw a decent sized picture — thanks!)

(f) Explain how to find $\int_{\partial\gamma} \alpha$ **without computation**. (It is OK to describe it geometrically if you don't know the value.)

5. (8 points) On **any** manifold M: if β is an exact r-form, and ω is a closed s-form, show that $\beta \wedge \omega$ is **exact**.

6. (12=4+8 points) Show that all line integrals of $\omega = P\,dx + Q\,dy + R\,dz$ are independent of path only if the value of the integral over any closed (piecewise C^1) path is zero. Use this and Stokes' theorem to obtain a condition on P, Q, R which is sufficient to show independence of the path. [Assume ω is defined on all of $\mathbb{R}^3$.]

7. (10 points) Let X, Y be smooth vector fields on a manifold M and let f, g be smooth functions on M and prove that $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X$.

8. (15=4+5+6 points) On $\mathbb{R}^3$, consider the 2-form $\alpha = y\,dx \wedge dz + \sin(xy)\,dx \wedge dy + e^x dy \wedge dz$ and the vector field $X = z\frac{\partial}{\partial y}$. Compute
(a) $d\alpha$

(b) $i_X \alpha$

(c) $L_X \alpha$
HINT: You might want to use: $L_X(\varphi) = i_X(d\varphi) + d(i_X\varphi)$, where the exercises I photocopied from Boothby show how i_X works.

9. (20 points) Define the Lie derivative for 2-forms by $L_X(\varphi) = i_X(d\varphi) + d(i_X\varphi)$, for vector fields by $L_X Y = [X,Y]$, and for functions by $L_X f = Xf$. Using the rules for the exterior derivative d, prove that L_X satisfies the Leibnitz-type relation

$$L_X(\varphi(Y,Z)) = (L_X\varphi)(Y,Z) + \varphi(L_X Y, Z) + \varphi(Y, L_X Z).$$

HINT 1: You can find how i_X works by looking in the photocopied exercises from Boothby.
HINT 2: You might want to remember Cartan's formula for 1 and 2-forms. Recall that we did this for 1-forms in class: $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$.

HAVE A GREAT HOLIDAY AND SEE YOU IN JANUARY!