MATH 25B TAKE HOME FINAL EXAM

9:00am Wednesday 10th May through 4:45pm Monday 15th May 2006.

INSTRUCTIONS—PLEASE READ CAREFULLY!!

- This exam consists of 8 questions. You must do all the questions. (Some questions have several parts.) The exam is out of a possible 110 points.
- The deadline to submit solutions is 4:45pm on May 15th. Failure to submit by this time will result in a grade of zero! Exams may be submitted in person to my office, or placed in an envelope and put in my (snail) mailbox outside of SC 325.
- During the take home exam period **YOU MAY NOT TALK TO ANYBODY ABOUT THE EXAM** (except me). This includes your classmates and any math majors!
- This is an open book exam. You may use any results that were proved in class, or as part of your reading or as part of your homework **provided you state them clearly**. If you use a text (other than Spivak) in your solutions, please reference it appropriately.
- To make the job of grading easier please:
    1. write your name on your solutions and staple all pages together
    2. write on one side of the paper and submit the problems in the order assigned
    3. write neatly!
- I will be available for consultations during the exam period. I can help you with the following sorts of problems:
    1. you don't understand the question
    2. you are stuck on how to get started
    3. you would like to discuss some of the material covered in class or from the homework
    4. you have done a computation and want to check the result.

  Note that I will give the same information out to everybody (especially for hints to get you started). Consultation times are listed on the course webpage. I can also answer some questions by email (allow 24 hours for an answer).

1. $(4 + 8 = 12 \text{ points})$

(a) Consider the set S of points in $\mathbb{R}^5$ defined by the two equations:

$$xu^2 + yzv + x^2 z = 3$$

$$xyv^3 + 2zu - u^2 v^2 = 2$$

Show there is a neighborhood of the point $(1, 1, 1, 1, 1) \in S$ and a differentiable function $h : \mathbb{R}^3 \to \mathbb{R}^2$ such that in the neighborhood, the point $(x, y, z, u, v) \in S$ where $h(x, y, z) = (u, v)$.

(b) Find $Dh(1, 1, 1)$. (Find the matrix of the Jacobian of h at $(1, 1, 1)$.)

2. (10 points)

Given positive real numbers $x_1, \ldots, x_n$, we define their arithmetic and geometric means as follows:

$$A.M. = \frac{x_1 + \cdots + x_n}{n}$$

$$G.M. = \sqrt[n]{x_1 \cdots x_n}$$

Use Lagrange multipliers to prove that the geometric mean is always less than or equal to the arithmetic mean. *Hint:* start by assuming $x_1 \cdots x_n = 1$. *(Note: you must use Lagrange multipliers to answer this question.)*

3. $(1 + 1 + 8 = 10 \text{ points})$

(a) State the implicit function theorem

(b) State the inverse function theorem

(c) Deduce the inverse function theorem from the implicit function theorem.

(Remark: in class we deduced the implicit function theorem from the inverse function theorem. In this question you are going the other way.)

4. (2+2+2+2+3+1=12 points)

In this question we compute the volume of n-dimensional unit ball, $B_n = B_n(1)$, where $B_n(r) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \le ||x|| \le r\} \subset \mathbb{R}^n$. In particular, we want to prove that

$$(*) \qquad v(B_{2n}) = \frac{\pi^n}{n!}, \qquad v(B_{2n+1}) = \frac{2^{n+1}\pi^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}.$$

4. continued...

(a) Use a change of variables to prove that $v(B_n(r)) = r^n v(B_n)$.

(b) Prove that $v(B_1) = 2$, $v(B_2) = \pi$ (for B_2 you can use polar coordinates).

(c) Prove the identity

$$\chi_{B_n}(x_1, \ldots, x_n) = \chi_{B_2}(x_1, x_2)\chi_{B_{n-2}(\sqrt{1-x_1^2-x_2^2})}(x_3, \ldots, x_n).$$

(Here χ_A is the characteristic function of $A \subset \mathbb{R}^n$: $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise.)

(d) Use polar coordinates to prove that

$$\int_{B_2} (1 - x_1^2 - x_2^2)^{(n-2)/2} dx_1 dx_2 = \frac{2\pi}{n}.$$

(e) Use Fubini's Theorem and parts (a), (c) and (d) to show that

$$v(B_n) = \frac{2\pi}{n} v(B_{n-2}).$$

(f) Now use induction to prove equations $(*)$.

5. $(2 + 2 + 3 + 3 + 1 + 2 + 2 + 6 + 4 + 1 = 26$ points)

The 3-cube $\gamma : [0, 1]^3 \to \mathbb{R}^3$ is given by $\gamma(u^1, u^2, u^3) = (x^1, x^2, x^3) = (u^1 \cos(2\pi u^2), u^1 \sin(2\pi u^2), u^3)$.

(a) Sketch the image of γ in $\mathbb{R}^3$. Please make your sketch a good size—postage stamp size pictures not accepted!

For the 2-form $\alpha = x^1 dx^2 \wedge dx^3 + x^2 dx^3 \wedge dx^1 + x^3 dx^1 \wedge dx^2$ find:

(b) $d\alpha$

(c) $\gamma^*(d\alpha)$

(d) $\gamma^*\alpha$

(e) $d(\gamma^*\alpha)$

(f) Calculate $\int_\gamma d\alpha$ **from the definition**.

(g) From your sketch in (a) above, why should you expect this answer? (I'm looking for a few sentences here, not an essay.)

(h) $\partial\gamma = \gamma_{1,1} +$??? Now find $\gamma_{1,1}(u^2, u^3) =$??? (That is, write out an expression for the map.) Do this for all $\gamma_{i,\alpha}$, $i = 1, 2, 3$, $\alpha = 0, 1$. Can you simplify the expression for $\partial\gamma$?

(i) Set up the integral $\int_{\partial\gamma} \alpha$.

(j) Calculate $\int_{\partial\gamma} \alpha$.

6. (10 points)
If only the surface were closed...

Let $\omega = z^2 x dy \wedge dz - (\frac{1}{3}y^3 + \tan z)dx \wedge dz + (x^2 z + y^2)dx \wedge dy$. Compute

$$\int_s \omega$$

where S is the top half of the sphere $x^2 + y^2 + z^2 = 1$ (including the boundary), oriented so that the boundary circle is oriented anti-clockwise when viewed from above.
...then we could use Stokes's Theorem.

7. (3+ 8+1=12 points) *The Mayer-Vietoris sequence*

Suppose that U and V are two open subsets of $\mathbb{R}^n$ and that $M = U \cup V$. Any differential form on $U \cup V$ restricts to give differential forms on U and V; similarly any differential form on U or on V restricts to give a form on $U \cap V$. So we have a diagram:

$$0 \longrightarrow \Omega^k(U \cup V) \xrightarrow{i_k} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j_k} \Omega^k(U \cap V) \longrightarrow 0$$

where $i_k(\omega) = (\omega|_U, \omega|_V)$ and $j_k(\phi, \eta) = \phi|_{U \cap V} - \eta|_{U \cap V}$.

(a) Let $\mathbb{R} = U' \cup V'$ where $U' = (-1, \infty)$ and $V' = (1, \infty)$. Show that any 0-form (i.e. smooth function) $f : U' \cap V' \to \mathbb{R}$ can be written as $g|_{U' \cap V'} - h|_{U' \cap V'}$ where $g : U' \to \mathbb{R}$ and $h : V' \to \mathbb{R}$ are smooth functions.

Hint: it may help to use a partition of unity. We know there exist smooth functions $\rho_{U'} : \mathbb{R} \to \mathbb{R}$ and $\rho_{V'} : \mathbb{R} \to \mathbb{R}$ such that $\rho_{U'} + \rho_{V'} = 1$ and $\rho_{U'}$ is zero outside U' and $\rho_{V'}$ is zero outside V'.

(b) Show that (i) the sequence above is exact and (ii) that there is a short exact sequence of complexes. You may assume anything you need about the existence of partitions of unity.

Hint: to show that j_k is surjective, it may be helpful to use/generalize part (a) of this question...

(c) Show that there is a long exact sequence of cohomology groups:

$$\begin{array}{l} 0 \longrightarrow H^0_{dR}(U \cup V) \longrightarrow H^0_{dR}(U) \oplus H^0_{dR}(V) \longrightarrow H^0_{dR}(U \cap V) \longrightarrow \\ \longrightarrow H^1_{dR}(U \cup V) \longrightarrow H^1_{dR}(U) \oplus H^1_{dR}(V) \longrightarrow H^1_{dR}(U \cap V) \longrightarrow \\ \longrightarrow H^2_{dR}(U \cup V) \longrightarrow H^2_{dR}(U) \oplus H^2_{dR}(V) \longrightarrow H^2_{dR}(U \cap V) \longrightarrow \dots \end{array}$$

8. (2+10+6=18 points) *Cohomology of* $\mathbb{R}^2$ *take away* n*-points.*

(a) Let B be a star-shaped open subset of $\mathbb{R}^2$. Compute the de Rham cohomology groups $H^i_{dR}(B)$.

(b) Let $A = \mathbb{R}^2 \setminus \{(0,0)\}$. By writing A as a union of star shaped open sets, compute the de Rham cohomology groups $H^i_{dR}(A)$. *Hint:* question 8 is useful.

(c) Compute the de Rham cohomology groups of the plane with n points removed. *Hint:* induction.

Conclusion: there is no smooth bijection between $\mathbb{R}^2 \setminus \{n \text{ points}\}$ *and* $\mathbb{R}^2 \setminus \{m \text{ points}\}$ *unless* $n = m$*!*

You've been a fantastic class. I wish you all the best in your future studies.

Have a great summer vacation!