

Math 25a Final Exam

Practice Questions

Last updated 16th Jan 2006

- (1) Go over all your old homework assignments!
- (2) Make sure you understand the main results from class and how to use them. Also make sure you understand the main techniques used in the proofs. There are a number of ideas that just keep coming up. (Looking at HW's also helps with both of these things.)
- (3) Fan, Shiyang and David have pointed out that there are some old Math 25 exams archived on the web. (Thanks guys!) They can be found at <http://www.fas.harvard.edu/~exams/>. Please note that the syllabus, texts and level of difficulty for Math 25 changes with each instructor. You'll find Laurent Berger's exams very difficult and Dick Gross' exams more straightforward. The Math 23 exams are also great for revision for both True/False questions and also practice for questions which involve calculations.

The following problems will help you see some things that might be similar to questions on the exam. They do not cover the exact same material!

- (4) Are the following TRUE or FALSE? Justify your answers. (Note this should be a sentence or two, not an essay.)
 - (a) If A and B are symmetric 2×2 matrices over \mathbb{R} , then AB is also symmetric.
 - (b) There are invertible 3×3 matrices A and S such that $SAS^{-1} = -A$
 - (c) If A is an $n \times n$ matrix with $\det(A) = 0$, then at least one column of A must be a scalar multiple of some other column of A .
 - (d) If A is an $n \times n$ matrix with n distinct real eigenvalues, then all eigenvalues of A^2 are non-negative.
 - (e) There are also more true/false questions in Otto Bretscher's linear algebra text on reserve in Cabot library.
- (5)
 - (a) Let W be a subspace of a finite dimensional vector space V . Let (w_1, \dots, w_m) be a basis for W . Show that this basis can be extended to form a basis of V .
 - (b) Let V be a finite dimensional vector space and let (v_1, \dots, v_k) be a list of vectors

in V . Let $W = \text{span}(v_1, \dots, v_k)$. Show that there is a sublist of (v_1, \dots, v_k) which is a basis for W .

(6) Suppose V is an n -dimensional inner product space over \mathbb{R} and that v is a nonzero vector in V and that Z is the subspace of V consisting of vectors perpendicular to v . What is the dimension of Z ? (Of course, you should prove your answer!)

(7) Compute

$$\begin{pmatrix} 3 & 0 & 2 \\ 0 & 0 & 0 \\ -4 & 0 & -3 \end{pmatrix}^{2006}.$$

You may, if you wish, use the fact that the eigenvalues of this matrix are $0, 1$ and -1 . (Note: you aren't supposed to use a calculator or a computer on this question!)

(8) Calculate the dimension (as a vector space over \mathbb{C} of the vector space

$$V = \{p : \mathbb{C} \rightarrow \mathbb{C} \mid p \text{ is a polynomial of degree at most } n \text{ such that } p(1) = p(2)\}.$$

(9) Suppose that W is a subspace of a finite-dimensional vector space V . Recall that the annihilator W^0 of W is defined to be

$$W^0 = \{h \in V^* \mid h(w) = 0 \text{ for all } w \in W\}.$$

(a) Show that $\dim(W) + \dim(W^0) = \dim(V)$.

(b) Corresponding to every element $f \in (V/W)^*$ there is an element $g \in V^*$; the linear functional g is defined by $g(v) = f(v + W)$. (Or $g(v) = f([v])$, where $[v]$ is the equivalence class of v in V/W .) Show that the correspondence $f \mapsto g$ defines an isomorphism between $(V/W)^*$ and W^0 . (Note: I have a feeling you've done something like this on your HW??)

(10) Let V be a vector space of dimension n and W a vector space of dimension m . Given linear transformations $S : V \rightarrow V$ and $T : W \rightarrow W$, define $S \oplus T : V \oplus W \rightarrow V \oplus W$ to be the linear transformation that maps (v, w) to $(S(v), T(w))$.

(a) Define the map $\rho : V \oplus W \rightarrow V \otimes W$ by $(v, w) \mapsto v \otimes w$. Show that there exists a unique linear transformation $S \otimes T : V \otimes W \rightarrow V \otimes W$ which makes the following diagram commute:

$$\begin{array}{ccc} V \oplus W & \xrightarrow{\rho} & V \otimes W \\ S \oplus T \downarrow & & \downarrow S \otimes T \\ V \oplus W & \xrightarrow{\rho} & V \otimes W. \end{array}$$

(Note: First you need to work out what $S \otimes T$ is. (Hint: use the universal mapping property of tensor products.) Then check that $(S \otimes T) \circ \rho = \rho \circ (S \oplus T)$.)

(b) What is the determinant of $S \otimes T$? Prove that your answer is correct. (Note: this question is hard and would only be worth a few points on an exam.)

(11) Consider the system of linear equations $Ax = b$, where

$$A = \begin{pmatrix} 6 & 0 & 2 & 0 \\ 0 & -3 & 0 & -6 \\ 2 & 1 & 1 & 2 \\ 1 & -2 & -1 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 18 \\ 3 \\ 6 \\ 1 \end{pmatrix}.$$

- (a) Find all solutions x of $Ax = b$.
- (b) Find the rank and nullity of A .

(12) Are the following TRUE or FALSE? Justify your answers. (Note this should be a sentence or two, not an essay.)

- (a) If V and W are subspaces of a vector space U , then so is $V \cap W$.
- (b) Any matrix $A \in \mathbb{R}^n$ is similar to a diagonal matrix. (Note: two $n \times n$ matrices B and C are *similar* if there is an invertible matrix S such that $B = S^{-1}CS$.)
- (c) If $A : V \rightarrow V$ is linear and $\ker(A)$ is nontrivial, then A is not surjective.
- (d) Any three non-zero vectors in \mathbb{R}^3 may be turned into an orthogonal basis for \mathbb{R}^3 via the Gram-Schmidt orthogonalization process.
- (e) If $u, v \in \mathbb{R}^n$ are orthogonal, then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.
- (f) If V is an inner product space and $u, v \in V$ satisfy $\langle u, v \rangle = 0$, then either $u = 0$ or $v = 0$.

(13) Suppose that V is a finite dimensional vector space over \mathbb{C} , and that the linear map $T : V \rightarrow V$ satisfies $T^k = I$ for some $k > 0$.

- (a) What are the possible eigenvalues for T ?
- (b) Show that T is diagonalizable.

(14) (a) Suppose that W is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V .

(b) Suppose that V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W .

(15) Let $V = \mathbb{R}^n$ and let U be a subspace of V . We define the orthogonal complement of U as:

$$U^\perp = \{v \in V \mid \langle v, u \rangle = 0 \forall u \in U\}.$$

- (a) Show that $V = U \oplus U^\perp$.

Thus each $v \in V$ can be written as $v = u + w$ with $u \in U$ and $w \in U^\perp$. Define a map $P_U : V \rightarrow V$ by the rule $P_U(v) = u$ with v and u as above.

- (b) Show that P_U is linear.
- (c) Show that $Im(P_U) = U$.
- (d) Show that $Ker(P_U) = U^\perp$.
- (e) Show that $\|P_U(v)\| \leq \|v\|$ for all $v \in V$.
- (f) Find the matrix of P_U (with respect to the standard basis) in the case when $V = \mathbb{R}^3$ and $U = span\{(1, 0, -1), (1, 1, 1)\}$.

(16) Let $V = \mathbb{R}^n$ and define $End(V) = \{T : V \rightarrow V \mid T \text{ is linear}\}$ to be the ring of endomorphisms of V .

Prove that if $T \in End(V)$, then there is some nontrivial polynomial p with $deg(p) \leq n^2$ that T “satisfies” in the sense that $p(T) = 0$, where 0 represents the zero linear transformation.

(HINT: This is really quite a straightforward question with some fancy words thrown in. You’ve seen $End(V)$ as $\mathcal{L}(V)$ in Axler. The word “ring” is an algebraic term that you can look up online. You already know that $\mathcal{L}(V, W)$ has an algebra structure on it - see Axler page 40 and 41. This knowledge is enough to do this question.)