Math 25a Homework 1

Due Tuesday 27th September 2005.

The first part of the homework contains problems from elementary set theory. Most (but not all) of the problems in this section are quite straightforward. The emphasis is to write proofs rigorously. What does this mean? Good proofs are:

- Correct — ideally, every statement should follow from axioms or from what has been proved before.
- Concise — a proof should not contain anything that is not necessary.
- Readable — Human beings both write and read proofs. Don't be afraid to explain in words what you are doing. For example, before embarking on a long computation, it is a good idea to explain what you are doing and why you are doing it.

1 Sets and Maps

(1) Let $A, B \subset X$. Prove that $A \subset B$ if and only if $X \smallsetminus B \subset X \smallsetminus A$.

Hint: This is an if and only if statement, so you have to prove both that the first statement implies the second and that the second statement implies the first.

(2) Prove the following statements (here A, B and C are three sets and $A \times B$ is defined to be the set of all pairs (a, b) with $a \in A$ and $b \in B$).

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(b) $A \smallsetminus (B \cup C) = (A \smallsetminus B) \cap (A \smallsetminus C)$

(3) Let $f : A \to B$ be a map, let W, X be two subsets of A and let Y, Z be two subsets of B. Determine whether the following are true or false.

(a) $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$

(b) $f(W \cap X) = f(W) \cap f(X)$

(c) If $Y \subset Z$ then $f^{-1}(Z \smallsetminus Y) = f^{-1}(Z) \smallsetminus f^{-1}(Y)$

Hint: This problem asks you to give a proof or a counterexample. If you are looking for a counterexample, first try looking for something simple — for example sets with only a few elements.

(4) Let $f : X \to Y$ be a map of sets. Prove that the following are equivalent:
(a) f is injective.
(b) $f^{-1}(t)$ contains at most one element for any $t \in Y$
(c) $f(A \cap B) = f(A) \cap f(B)$ for any $A, B \subset X$.
Hint: it suffices to prove $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

(5) Suppose that A and B are two sets, and denote by A^B the set of all maps $f : B \to A$. Construct a bijection between $A^{B \times C}$ and $(A^B)^C$.

(6) Prove that if X and Y are two sets such that there are injective maps $f : X \to Y$ and $g : Y \to X$, then there is a bijection from X to Y.

Hint: This is an example of a result that is easy to state but requires thought to prove. There are two hints to help you with this question, available upon request.

2 Equivalence relations

(1) Given a set A and an equivalence relation $\sim$, recall from class that an *equivalence class* $[a]$ is defined as $[a] := \{b \in A : b \sim a\}$. Prove that if $[a], [b]$ are two equivalence classes, then either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

(2) Let A be the set of all pairs of integers (a, b) such that $b \neq 0$. Define $(a, b) \sim (c, d)$ if and only $ad = bc$.

(a) Prove that this is an equivalence relation.

Definition: The quotient of A by the equivalence relation $\sim$ is the set of all equivalence classes: $A/\sim := \{[a] : a \in A\}$.

(b) In class, we saw that $A/\sim$ looks a lot like the rational numbers. We defined addition on $A/\sim$ to be $[(a, b)] + [(c, d)] = [(ad + bc, bd)]$. Check that this definition is well-defined. (That is, if $[(a, b)] = [(a', b')]$ and $[(c, d)] = [(c', d')]$, then $[(ad+bc, bd)] = [(a'd' + b'c', b'd')]$.)

(c) Define multiplication ($[(a, b)] \cdot [(c, d)] =?$) and check that it is well defined.

(d) Prove the existence of additive inverses.

(Hint: To do this you will first need to name the additive identity correctly.)

(e) This part is **not** to be handed in! Convince yourself that $A/\sim$ with addition and multiplication defined as in parts (b) and (c) obeys the rest of the field axioms.

(3) Let $f : S \to T$ be a map of sets. Define a relation on S by $s_1 \sim s_2$ if and only if $f(s_1) = f(s_2)$.

(a) Prove that this is an equivalence relation

(b) Find an injective map $\tilde{f} : S/\sim \to T$.

Hint: first construct the map and prove that it is injective.

3 Combinatorics

(1) Prove by induction that the following two statements are true for every positive integer n.

(a) $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$

(b) The number $2^{n+2} + 3^{2n+1}$ is a multiple of 7.

(2) Show that if A and B are non-empty finite sets, then $Card(A^B) = Card(A)^{Card(B)}$.
(Optional: Think about why this is true if one of A or B is empty.)

(3)(a) A string of left and right brackets is said to be balanced if each left bracket has (to its right) a matching right bracket. How many balanced strings of n left and n right brackets are there? (I'm only looking for answers for $n = 0, 1, 2, 3, 4$ and maybe 5.)

(b) For any positive integer n, evenly distribute $2n$ points on the circumference of a circle. In how many ways can these $2n$ points be paired off as n chords, where no two chords intersect? (I'm only looking for answers for $n = 1, 2, 3, 4$ and maybe 5.)

(c) A railway track consists of a left track, a right track both of which merge into a central track. There are n railway wagons on the left track. The wagons are moved from the left track to the right track through the central track. It is assumed that the central track can accommodate all n of them and that they travel only from left to right (that is, they may not be moved from the central track back to the left track). How many different ways are there to move the n wagons from the left track to the right track? (I'm only looking for answers for $n = 0, 1, 2, 3, 4$ and maybe 5.)

(d) Hmmm, do the numbers from parts (a), (b) and (c) look familiar? Can you show that these problems are equivalent? (For example, given a bracketing, can you get a chord diagram and vice-versa?)

(4) We defined $\binom{n}{k}$ to be number of subset of $\{1, 2, 3, \ldots, n\}$ with k elements. By constructing bijections between appropriate sets, prove that:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Show also that

(a)

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{k-1}{k-1}$$

(b)

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

4 Fields and other things

(1) Let F be a field. Prove that the axioms for multiplication imply the following statements:

(a) If $x \neq 0$ and $xy = x$ then $y = 1$.
(b) If $x \neq 0$ and $xy = 1$ then y is uniquely determined.
(c) If $x \neq 0$ and $xy = xz$ then $y = z$.
(d) If $x \neq 0$ then $1/(1/x) = x$.

(2) Prove that $\mathbb{Z}/p\mathbb{Z}$ is a field if and only if p is a prime number.

(3) If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

(4) Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$.

(5)(a) Let $z_k = \cos(\frac{2\pi k}{n}) + i\sin(\frac{2\pi k}{n}) \in \mathbb{C}$. Show that the polynomial $x^n - 1$ factors as follows:

$$x^n - 1 = \prod_{k=0}^{n-1}(x - z_k).$$

(b) A regular n-gon is inscribed in a unit circle. Label the vertices of the n-gon $v_1, v_2, \ldots, v_n$. Pick a vertex, say v_1, and take all line segments from v_1 to the rest of the vertices. Find the product of the lengths of these $(n-1)$ segments.

5 Just for Fun

I am a Sudoku addict. If you don't know what a Sudoku puzzle is, then go to the following website to find out.

http://www.dailysudoku.co.uk/sudoku/index.shtml

Another website I quite like is:

http://www.sudokufun.com/

Of course I'm not the only one who likes Sudoku puzzles. (This includes my own non-mathematical parents.) Perhaps many of you are addicted as well. I know that many papers (ex. UK Daily Telegraph, USA New York Times or the Australian Sydney Morning Herald) now have daily Sudoku puzzles. Enjoy!