Math 25a Homework 11

Due Tuesday 6th December 2005.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) (a) Problem 6 on page 94 of Axler.
(b) Problem 8 on page 94 of Axler.

(2) (a) Problem 18 on page 95 of Axler.
(b) Problem 19 on page 95 of Axler.

(3)(a) Problem 21 on page 95 of Axler.
(b) Problem 22 on page 95 of Axler. (Hint: see page 92 for notation $p_{U,W}$.)

(4) Problem 5 page 122 Axler.

(5) Problem 10 page 123 of Axler. (Please make a note of your answer, you'll need this for HW 12 next week!)

(6) Problem 17 on page 124 of Axler.

2 Ivan's problems

(1) *Products and coproducts of vector spaces*
Let V_i be a vector space over a field F. For each $i \in I$ where I is some indexing set (possibly infinite) we define

$$\prod_{i\in I} V_i = \left\{ f : I \to \bigcup_{i\in I} V_i \mid f(i) \in V_i \ \forall i \right\},$$

$$\coprod_{i\in I} V_i = \left\{ f : I \to \bigcup_{i\in I} V_i \mid f(i) \in V_i \text{ and } f(i) = 0 \in V_i \text{ for all but finitely many } i \right\}.$$

We put a vector space structure on $\prod_{i\in I} V_i$ as follows. If $f_1, f_2 \in \prod_{i\in I} V_i$ and $\alpha \in F$, then define $f_1 + f_2 \in \prod_{i\in I} V_i$ by $(f_1 + f_2)(i) = f_1(i) + f_2(i)$ and define $\alpha f_1 \in \prod_{i\in I} V_i$ by $(\alpha f_1)(i) = \alpha f_1(i)$.

(a) Check that $\prod_{i\in I} V_i$ is a vector space. In fact, just check any two of the vector space axioms and submit for grading. (Check the rest, but don't submit for grading.)

(b) Let $I = \{1, 2, \ldots, n\}$ and $V_i = F$ for all i. Show that $\prod_{i\in I} V_i \cong F^n$. (Do this by constructing a map from $\prod_{i\in I} V_i$ to F^n, checking it is linear and that it is invertible.)

(c) Check that $\coprod_{i\in I} V_i$ is a subspace of $\prod_{i\in I} V_i$.

(d) Show that when I is finite that $\prod_{i\in I} V_i = \coprod_{i\in I} V_i$ and that when I is infinite that $\coprod_{i\in I} V_i \subsetneqq \prod_{i\in I} V_i$. (For the latter part you should also assume that $V_i \neq \{0\}$ for all $i \in I$.)

(3) *Universal mapping properties*

Let V_i be a vector space over a field F and $\prod_{i\in I} V_i$ defined as in (1).

(a) Define a family of linear transformations as follows: let $proj_i : \prod_{i\in I} V_i \to V_i$ be defined by $proj_i(f) = f(i)$ for all $i \in I$.

Show that given *any* vector space W and *any* family of linear transformations $\gamma_i : W \to V_i$ (for all $i \in I$) there exists a unique linear transformation $\gamma : W \to \prod_{i\in I} V_i$ such that

$$proj_i \circ \gamma = \gamma_i \quad \forall i \in I. \qquad (\#)$$

(Hint: we need to define γ. So for $w \in W$ observe that $\gamma(w) \in \prod_{i\in I} V_i$, that is $\gamma(w)$ is a map. What does it do? Hint: define $\gamma(w)(i) := \gamma_i(w)$ for all $i \in I$. Now check γ is linear, check uniqueness and check property (#).)

The conditions described in part (a) are known as the universal mapping property (UMP) for products. Note: the tuple

$$\left(\prod_{i\in I} V_i \ , \ \{proj_i : \prod_{i\in I} V_i \to V_i\}_{i\in I} \right)$$

is called a *product* of the family $\{V_i\}_{i\in I}$. In part (b) we'll show that this UMP determines products uniquely up to isomorphism.

(b) Suppose that

$$\left(Z \ , \ \{p_i : \prod_{i\in I} V_i \to V_i\}_{i\in I} \right)$$

satisfies the UMP for products. (To repeat, this means that given any vector space W and any family of linear maps $\gamma_i : W \to V_i$ then there exists a unique map $\gamma : W \to Z$ such that $p_i \circ \gamma = \gamma_i$.) Show that $Z \cong \prod_{i\in I} V_i$.

(Hint: Use the UMP for Z and also for $\prod_{i\in I} V_i$ to construct linear transformations $\gamma_1 : Z \to \prod_{i\in I} V_i$ and $\gamma_2 : \prod_{i\in I} V_i \to Z$. Now show $\gamma_1 \circ \gamma_2 = I_{\prod V_i}$ and show $\gamma_2 \circ \gamma_1 = I_Z$. (Note that I_Z is the identity map on Z.) For this last part you will need to use the UMP again, in particular the part about uniqueness.)

(4) *Universal mapping properties continued*
Let V_i be a vector space over a field F and $\coprod_{i\in I} V_i$ defined as in (1).

Define a family of linear transformations as follows: for all $i \in I$ let $inj_i : V_i \to \coprod_{i\in I} V_i$ be defined by $inj_i(v_i) = f_{v_i}$ where f_{v_i} is the map defined by

$$f_{v_i}(j) = \begin{cases} v_i & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

You can show that given any W and a family of linear transformations $\gamma_i : V_i \to W$, there is a unique map $\gamma : \coprod_{i\in I} V_i \to W$ satisfying $\gamma \circ inj_i = \gamma_i$ for all $i \in I$. Namely $\gamma(f) = \sum_{i\in I} \alpha_i(f(i))$. (Note that this sum is well defined as $f(i) = 0$ for all but a finite number of i.)

This is the universal mapping property for coproducts. The tuple

$$\left(\coprod_{i\in I} V_i \ , \ \{inj_i : V_i \to \coprod_{i\in I} V_i\}_{i\in I} \right)$$

is called the coproduct of the vector spaces V_i. Note that the analogue of 2(b) holds — that is, the UMP determines the coproduct up to isomorphism. There is no real exercise for you to do here or to submit. Just think all of this through, make sure you understand what is going on and see if you can imitate the arguments in question 2 in this new setting.

(5) (a) In class we defined $V = V_1 \oplus V_2$ for subspaces V_1 and V_2 of the vector space V. Show that $V_1 \oplus V_2 \cong \coprod_{i\in I} V_i$.

(b) Now consider $W = \coprod_{i\in\{1,2\}} V_i$. Define $W_i \subset W$ by

$$W_i := \{f \in \coprod_{j\in I} V_i \mid f(j) = 0 \text{ for } j \neq i\}$$

Show (i) W_i is a subspace of W for all i (ii) $W_i \cong V_i$ and (iii) $W = W_1 \oplus W_2$.

(A word of caution: For vector spaces we often abuse notation and write $\bigoplus$ and $\coprod$ interchangeably.)

3 Warm up and Extra Problems.

As ever any of the exercises in Axler are good practice. In chapter 5 you might take a look at 1, 2, 3, 5, 7, 10, 11 and 12. In chapter 6 take a look at 21, 22, 23 and 26.

A *very* challenging exercise is problem 8 from chapter 6. You'll actually need to do problems 6 and 7 first! This is a fantastic question to really get you thinking about inner products. This question has several components. First you construct the inner product for $\mathbb{R}$, then for $\mathbb{C}$. Even doing so for $\mathbb{R}$ is hard. In particular homogeneity in the first slot requires a lot of thought. (You prove it in stages. Do it first for positive integers, then for positive rationals, then for all rationals, then for all reals!)

4 Just for fun!

So you all think multiple choice exams are easy — right? Well take a look at this one! (It's a lot of fun.)
http://www.math.wisc.edu/∼propp/srat-Q

You can find more information about this puzzle at:
http://www.math.wisc.edu/∼propp/srat.html