Math 25a Homework 2

Due Tuesday 4th October 2005.

1 Injections, surjections, bijections

(1) Prove that the composition of two injective functions is an injective function and that the composition of two surjective functions is a surjective function.

(2) Let $f : A \to B$ be a function. Show that the following are equivalent.

(a) There exists a function $g : B \to A$ such that $g(f(x)) = x$ for all $x \in A$ and $f(g(y)) = y$ for all $y \in B$.

(b) f is a bijection.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that f is *strictly increasing* if $f(x) < f(y)$ whenever $x < y$.

(a) Show that if f is strictly increasing, it is an injection. Must it be a surjection?

(b) Suppose f is a bijection and $f(0) = 0$ and $f(1) = 1$. Does it follow that f is strictly increasing?

(4) Give an explicit bijection between each of the following sets (see Rudin page 31 for the notation):

(a) $[1, 2]$ and $[3, 7]$

(b) $(0, 1)$ and $(0, \infty)$

(c) $[0, 1]$ and $[0, 1)$

(d) $\mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ *(Hint: see decimal expansions ...)*

2 Fields, rational and irrational numbers

(1) Given $x \in \mathbb{R}$ with $x > 0$ and an integer $k \geq 2$, define $a_0, a_1, \ldots$ recursively by setting $a_0 = \lfloor x \rfloor$ (this means the largest integer which is less than or equal to x) and a_n to be the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \ldots + \frac{a_n}{k^n} \leq x$$

(a) Show that $0 \leq a_i \leq k - 1$ for each $i \geq 1$.

(b) Let $r_n = a_0 + \frac{a_1}{k} + \ldots \frac{a_n}{k^n}$. Show that $\sup\{r_0, r_1, \ldots\} = x$.

(Note: The expression $r = a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \ldots$ is called the base-k expansion of $x \in \mathbb{R}$. If $k = 10$, this is the decimal expansion of x. The next part of the problem shows that the base-k expansion of x is almost unique.)

(c) Show that if we have sequences $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ such that

(I) $0 \leq a_i \leq k-1$ and $0 \leq b_i \leq k-1$

(II) $a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + \ldots = b_0 + \frac{b_1}{k} + \frac{b_2}{k^2} + \ldots$

(III) for each $N > 0$ there exists and $n > N$ and $m > N$ such that $a_n \neq k-1$ and $b_m \neq k-1$

then $a_i = b_i$ for all i.

(Hint: This last condition just says that neither sequence ends with an infinite sequence of $(k-1)$'s. You may need to use the fact that if $0 \leq |x| < 1$, then $1 + x + x^2 \ldots = \frac{1}{1-x}$.)

(2) Use the elementary order axioms to prove the following properities:

(a) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.

(b) If $x, y \geq 0$ and $x^2 > y^2$, then $x > y$.

(c) $\mathbb{R}$ is not bounded above.

(3) (a) Prove that there is exactly one way to make $\mathbb{Q}$ into an ordered field.

Hint: Assume that there is another order $x \prec y$ on $\mathbb{Q}$ satisfying the axioms for an ordered field and show that $x \prec y$ if and only if $x < y$.

(b) Let p be prime. Show that the field $\mathbb{Z}/p\mathbb{Z}$ cannot be made into an ordered field. Show that $\mathbb{C}$ cannot be made into an ordered field.

(4) (a) Show that $\sqrt{2} + \sqrt{3}$ is irrational.

(b) If a and b is irrational, must $a + b$ be irrational?

(5) Show that between any two distinct real numbers x, y, there is an irrational number.

(6) Rudin pg 22 q. 6 (This is an exercise to verify that indices behave how you would expect them to behave. That is, you know how indices work when they are integers, what about when they are rational and real numbers?)

(7) Rudin pg 22 q. 7 (This exercise uses what you've learned in q. 6 to define logarithms.)

3 Combinatorics and Countability

(1) Verify that

(a) $Card(\mathbb{N}) = Card(\mathbb{N}^2)$

(b) $Card(\{0,1\}^{\mathbb{N}}) = Card(\mathbb{R})$

(2) Let A, B, X be sets with $Card(A) = Card(B)$. Then show that $Card(A^X) = Card(B^X)$ and $Card(X^A) = Card(X^B)$.

(Hint: Recall that A^B is the set of all functions from B to A.)

(3) Let X be an infinite set.

(a) Show that for each positive integer n, there is a subset A_n of X with size n.

(b) Show that there is a countable subset of X. (Note: this implies that "countability" is the smallest possible infinite set.)

(Hint: Go look up the well-ordering theorem (equivalent to the axiom of choice).)

(c) Hence or otherwise show that any subset S of $\mathbb{N}$ is at most countable.

(4) Let S be a non-empty set. Show that the following are equivalent:

(a) S is at most countable (see Rudin p. 25 for a definition).

(b) there exists an injection $f : S \to \mathbb{N}$.

(c) there exists a surjection $g : \mathbb{N} \to S$.

Now define a surjective map from $\mathbb{N} \to \mathbb{Q}$ and hence show that the rational numbers are countable.

(5) Let X be an uncountable set and Y a countable subset of X. Show that $Card(X \setminus Y) = Card(X)$.

(Hint: first show that if A is an uncountable set, B is a countable set and disjoint from A (that is $A \cap B = \emptyset$) then $Card(A \cup B) = Card(A)$.)

(6) A real number x is called *algebraic* if it is the root of an equation $f(x) = 0$ where

$$f(x) = a_n x^n + \ldots + a_1 x + a_0$$

is a polynomial such that the coefficients $a_0, \ldots, a_n$ are integers. Numbers that are not algebraic are called *transcendental*.

(a) Show that the set of algebraic numbers is countable.

(b) Show that at least one transcendental number exists. Can you construct a bijection between the set of algebraic numbers and the set of transcendental numbers?

4 Just for Fun - not to be handed in.

A group of Math 25 students and their instructor play a game. The instructor places a cap with either a gold star or a blue star on the head of each student. The students sit in a circle and can see the stars on the caps of the other students but can't see their own. During the game the students are **not** allowed to communicate the type of star on another students cap.

The game begins when the instructor announces that she placed a cap with a gold star on at least one persons head. Every 3 minutes thereafter she asks whether or not anyone has worked out what the type of star is on their cap? If a student has figured it out, they will stand up. (Note: being highly truthful and very good at logic, a student will only stand up if they know for certain what type of star is on their cap.)

After exactly 10 rounds of the game (30 minutes) several students stand up. How many students were wearing caps with gold stars?