Math 25a Homework 3

Due Wednesday 12th October 2005.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from sections 1 and 2 separately from the problems in sections 3, 4 and 5. (So you'll hand in two stapled bundles of solutions for HW 3, each with your name on it and one with sections 1 and 2 and the other with sections 3, 4 and 5.)

1 Metric spaces

(1) (a) Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^k$. Find the value of t that minimizes $|\vec{x} + t\vec{y}|$. (So, just to be very clear, I'm using some notation here to distinguish between $x \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^k$. That is $\vec{x} = (x_1, x_2, \ldots, x_k)$, where each $x_i \in \mathbb{R}$.)

(b) Deduce that $|\vec{x} \cdot \vec{y}| \leq |\vec{x}||\vec{y}|$ for all $\vec{x}$, $\vec{y} \in \mathbb{R}^k$.

(c) Prove the *triangle inequality:* $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$ for all $\vec{x}, \vec{y} \in \mathbb{R}^k$.

(d) Hence or otherwise, show that if we define $d(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}|$, then $(\mathbb{R}^k, d)$ is a metric space.

Hint: The definitions of $\vec{x} \cdot \vec{y}$ and $|\vec{x}|$ are on page 16 of Rudin.

(2) Problem 10 on page 44 of Rudin.

(3) Problem 11 on page 44 of Rudin.

(4) *This problem introduces the p-adic topology on $\mathbb{Q}$. In thinking about this quesiton, it might be helpful to think of a particular p, the 3-adic topology for example.*
Given a non-zero rational number r, we can write it uniquely in the form

$$r = \frac{p^\nu n}{d}$$

where n and ν are integers, d is a positive integer, and neither n nor d is divisible by p. Define $\nu(r)$ to be the integer ν occurring in this expression. For $x, y \in \mathbb{Q}$, define

$$d_p(x, y) = \begin{cases} p^{-\nu(x-y)} & \text{if } x \neq y \\ 0 & \text{if } x = y. \end{cases}$$

(a) Show that $(\mathbb{Q}, d_p)$ is a metric space, and that in fact $d(x, z) \leq \max(d(x, y), d(y, z))$.

(b) Show that if $x \in N_r(a)$, then $N_r(x) = N_r(a)$, so that any point of the neighborhood $N_r(a)$ is a "center" of that neighborhood.

(c) Show that given two neighborhoods $N_{r_1}(a_1)$ and $N_{r_2}(a_2)$, either they are disjoint or one is contained in the other.

This metric space is weird huh?

2 Open sets, closed sets

(1) Problem 6 on page 43 of Rudin.

(2) Problem 7 on page 43 of Rudin.

3 More open sets, closed sets

(3) Problem 9 parts (a) through (d) on page 43 of Rudin.

(4) Problem 29 on page 45 of Rudin.

4 Compactness

(1) Problem 12 on page 44 of Rudin.

(2) Problem 13 on page 44 of Rudin.

(3) Problem 16 on page 44 of Rudin.

5 Supplementary Problems

These are some other cool problems to think about. You don't need to hand these problems in, your grade will not be affected at all.

(1) Give an example of a subset E of $\mathbb{R}$ such that the following sets are all different:

$$E, \quad \overline{E}, \quad \mathring{E}, \quad \mathring{\overline{E}}, \quad \overline{\mathring{E}}, \quad \overline{\mathring{\overline{E}}}, \quad \mathring{\overline{\mathring{E}}}.$$

(2) Product Topology. Let (E_1, d_1) and (E_2, d_2) be two metric spaces. Define a distance on $E_1 \times E_2$ by

$$d((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2).$$

Show that $U \subset E_1 \times E_2$ is open if and only if for each point $(u_1, u_2) \in U$, there exist U_1 and U_2 open in E_1 and E_2 respectively, with $(u_1, u_2) \in U_1 \times U_2$ and $U_1 \times U_2 \in U$.

6 The Math Puzzler - just for fun!

Each week there will be a "math puzzler" for the class to think about. Please feel free to submit a"puzzler" you think the class might enjoy. The "puzzlers" don't have to be difficult, nor related to the material in class — they just have to be fun to think about!

The hungry worm. There is a big cube made up of 27 little cubes arranged in a $3 \times 3 \times 3$ fashion. (So think about a rubik's cube and you'll have the right idea.) There is a hungry worm who wants to eat the big cube by eating the little cubes one at a time. This worm is very fussy about how he likes to eat the little cubes. Once he finishes eating a little cube, he'll only move on to eating the next one by eating across the face of the cube. He never eats across an edge nor a vertex. So our worm starts eating the big cube from the outside, munching one little cube and then the next. Can our worm ever end up eating the center little cube last? [Optional: What about other sizes of big cubes (eg. $4 \times 4 \times 4$ or $5 \times 5 \times 5$ etc)?]