Math 25a Homework 5

Due Tuesday 25th October 2005.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

I've included three optional sections on this HW. The first is full of "warm up" problems, the second full of harder supplementary problems and the last the math puzzler as per usual. So you **must** do the problems in Sections 1 and 2, but the problems in Sections 3, 4 and 5 are completely optional.

1 Alison's problems

(1) Find an example of a sequence $\{p_n\}$ in $\mathbb{R}$ such that (i) $\{p_n\}$ is bounded (ii) $|p_{n+1} - p_n| \to 0$ as $n \to \infty$, but which is still not convergent.

(2) Problem 3 on page 78 of Rudin.

(3) Problem 5 on page 78 of Rudin.

(4) Let $\{p_n\}$ be any bounded sequence of real numbers and let $l = \liminf p_n$ and $u = \limsup p_n$. Prove that $l \leq u$ and moreover, that $l = u$ if and only if $\lim_{n\to\infty} p_n$ exists, in which case $\lim_{n\to\infty} p_n = l = u$.

(Hint: Recall that $\liminf p_n = \lim_{n\to\infty} (\inf\{p_n, p_{n+1}, \dots\})$ *and* $\limsup p_n$ *is defined similarly.)*

(5) Problem 12 on page 79 and 80 of Rudin.

2 Ivan's problems

(1) Problem 7 on page 78 of Rudin.

(2) This question is set in $\mathbb{R}$. Let $\alpha > 0$ and let x_0 arbitrary and define a sequence x_n recursively by

$$x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}).$$

Show that $x_n \to \sqrt{\alpha}$. (Note this is Newton's method for solving $x^2 - \alpha = 0$.)

(3) Problem 20 on page 8 of Rudin.

(4) Problem 21 on page 82 of Rudin.

(5) Problem 22 on page 82 of Rudin.

3 Warm-up Problems

These are some optional problems that are here as a "warm-up". They are meant to be straightforward questions to help you think about the definitions and material covered.

(1) Let $\{p_n\}$ be a sequence of real (or complex) numbers and let $p \in R$ (or $\mathbb{C}$). Show that the following are equivalent.

(a) $p_n \to p$.
(b) $p_n - p \to 0$.
(c) $|p_n - p| \to 0$.

(2) Show that if (p_n) is a convergent sequence of real (or complex) numbers, then $|p_{n+1} - p_n| \to 0$.

(3) *The Squeezing Lemma* Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three sequences of real (or complex) numbers such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Suppose that $a_n \to l$ and $c \to l$ (the same l), then prove that $b_n \to l$ as well.

4 Supplementary Problems

Here are some harder optional problems for you to think about.

(1) Problem 24 page 82 of Rudin gives you the recipe to complete an arbitrary metric space. Problem 25 on page 82 of Rudin asks you to think in particular about completing $\mathbb{Q}$ to $\mathbb{R}$. Note that we already did this in class using Dedekind cuts. Think for yourself how you might use Rudin's recipe to construct $\mathbb{R}$ from $\mathbb{Q}$ using Cauchy sequences. (Note, you'll need to define addition, multiplication etc.)

(2) Now think about $\mathbb{Q}$ with the p-adic metric (see HW 3). A (much) harder problem is to prove that $(\mathbb{Q}, d_p)$ is not complete. The completion of $\mathbb{Q}$ with respect to this metric is called the $\mathbb{Q}$-adic numbers. These play a very important role in number theory.

(3) Go hunt in the library or online and find a proof that e is transcendental. What about π — can you find proofs for irrationality or transcendental nature of π?

5 The Math Puzzler - just for fun!

Each week there will be a "math puzzler" for the class to think about. Please feel free to submit a "puzzler" you think the class might enjoy. The "puzzlers" don't have to be difficult, nor related to the material in class — they just have to be fun to think about!

This problem is from Ivan Corwin.

Given a 5×5 grid of dots, can you draw a closed curve, only using horizontal or vertical lines, and not lifting your pencil, such that every dot is crossed once and such that you end at your starting point? Can you generalize this problem? That is, for what $m \times n$ is this impossible?

Now consider a $5 \times 5 \times 5$ array of dots with the same rules (only go up, down, left right, in, out and go through every dot only once and end at the starting point). Is is possible to draw a closed curve with these rules? For what size three dimensional arrays is it impossible? Generalize to all dimensions.