Math 25a Homework 8

Due Tuesday 15th November 2005.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) (a) If L, M, and N are subspaces of a vector space V, show that $L \cap (M + N) = (L \cap M) + (L \cap N)$ does not necessarily hold.
(b) Prove that $L \cap (M + (L \cap N)) = (L \cap M) + (L \cap N)$.

(2) Let S be a subset of a vector space V. Show that the following are equivalent:
    (a) S is a basis for V.
    (b) S is a maximal linearly independent subset of V.
    (c) S is a minimal spanning set in V.

(3) Problem 10 on page 35 of Axler.

(4) Problem 14 on page 36 of Axler.

(5) (a) Find a statement of Zorn's Lemma.
    (b) Use Zorn's Lemma to show that every vector space contains a maximal linearly independent subset S. Note that from 2(b) above, this tells you that S is a basis for V.

2 Ivan's problems

(1) Problem 15 on page 36 of Axler.

(2) Problem 17 on page 36 of Axler.

(3) Let V be a vector space over the field F and let W be a subspace of V. Define a relation $\sim$ in V by: $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$.

(a) Prove that $\sim$ is an equivalence relation.

The set of all equivalence classes is denoted V/W — the quotient space of V with respect to W. Let $[v_1]$ denote the equivalence class of v_1. Now we show V/W is also a vector space in the following way:

(b) Define addition on V/W and show that it is well defined. That is $[v_1] + [v_2] =?$ and show that it is independent of the representatives you have chosen.

(c) Define scalar multiplication and show that it is well defined. That is $a \cdot [v_1] =?$ (where $a \in F$), and show that it is independent of the representatives you have chosen.

(d) Now choose **two** vector space axioms and check that they hold. Submit your proofs. Convince yourself that all the axioms hold as well (but don't worry about submitting these extra proofs).

(4) Suppose that for each $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, the function f is defined by:
   (a) $f(x) = x_1 + x_2$,
   (b) $f(x) = x_1 - (x_2)^2$,
   (c) $f(x) = x_1 + 1$,
   (d) $f(x) = 2x_2 - 3x_3$.
In which of these cases is f a linear functional?

(5) Supose that for each x in $\mathcal{P}(\mathbb{R})$ (polynomials with real coefficients, in say the variable t), the function f is defined by:
   (a) $f(x) = \int_{-1}^{+2} x(t)dt$,
   (b) $f(x) = \int_0^2 (x(t))^2 dt$,
   (c) $f(x) = \frac{dx}{dt}$,
   (d) $f(x) = \frac{dx}{dt}|_{t=1}$.
In which of these cases is f a linear functional?

3   Warm up and Extra Problems.

There are HEAPS of great linear algebra problems. If you are new to linear algebra, you will find that Otto Bretcher's text is very helpful. (Details about this can be found on the course webpage.) This week problems 8 and 9 on page 35 of Axler get you to think about bases. Problems 11 and 16 on page 36 of Axler are also great for understanding subspaces and dimension.

4   Just for fun!

This week I'll give the math puzzler a break, but I have found two things that you might enjoy. However please feel free to submit a "puzzler" you think the class might enjoy. The

"puzzlers" don't have to be difficult, nor related to the material in class — they just have to be fun to think about!

(1) A tongue in cheek "Declaration of Linear Independence".
See http://www-math.bgsu.edu/%7Egrabine/linear.html

(2) An a capella group called "The Klein Four Group" is based at the Northwestern University mathematics department. Their song "Finite simple group (or order two)" is a particular favorite of mine.
See http://www.math.northwestern.edu/∼matt/kleinfour/