

# Math 25b Homework 11

Due Wednesday 26th April 2006.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

## 1 Alison's problems

(1) *The wedge product is supercommutative*

Let  $V$  be an  $n$ -dimensional vector space, let  $\omega \in \Omega^k(V)$  and let  $\eta \in \Omega^l(V)$ .

(a) Show that

$$\eta \wedge \omega(v_1, \dots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma) \omega(v_{\tilde{\sigma}(1)}, \dots, v_{\tilde{\sigma}(k)}) \eta(v_{\tilde{\sigma}(k+1)}, \dots, v_{\tilde{\sigma}(k+l)})$$

where

$$\begin{aligned} \tilde{\sigma}(1) &= \sigma(l+1) & \tilde{\sigma}(k+1) &= \sigma(1) \\ \tilde{\sigma}(2) &= \sigma(l+2) & \tilde{\sigma}(k+2) &= \sigma(2) \\ & \vdots & \vdots & \\ \tilde{\sigma}(k) &= \sigma(l+k) & \tilde{\sigma}(k+l) &= \sigma(l) \end{aligned}$$

(b) Find a permutation  $\pi$  such that  $\tilde{\sigma} = \sigma\pi$ .

(c) Deduce a relationship between  $\operatorname{sgn}(\sigma)$  and  $\operatorname{sgn}(\tilde{\sigma})$  and show that

$$\eta \wedge \omega = (-1)^{kl} \omega \wedge \eta.$$

(2) *Wedge products and determinants*

(a) Problem 4–1 on page 84 of Spivak

(b) State and prove a relationship between  $\phi_{i_1} \wedge \dots \wedge \phi_{i_k}$  and  $\operatorname{Alt}(\phi_{i_1} \otimes \dots \otimes \phi_{i_k})$ .

(This fills in a detail in the proof that  $\{\phi_{i_1} \wedge \dots \wedge \phi_{i_k} : 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$  is a basis for  $\Omega^k(\mathbb{R}^n)$ .)

(3) A vector field  $F$  is called *conservative* if and only if  $F = \text{grad}(f)$  for some function  $f$ . (For a reminder about  $\text{grad}(f)$  see problem 4–18 page 96 of Spivak.) Is the following vector field in  $\mathbb{R}^3$  conservative?

$$F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2y \\ 2y \\ 3x^2 \end{pmatrix}$$

If it exists,  $f$  is called the *potential function* for  $F$ .

(4) *Curl*

Consider the solid cylinder in  $\mathbb{R}^3$  given by

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq 1\}$$

and the vector field  $\mathbf{v}$  on  $S$  given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(a) Sketch the vector field  $\mathbf{v}$

Note that  $\mathbf{v}$  is the velocity field produced by rotating the (rigid) cylinder  $S$  about the  $z$ -axis with constant angular speed  $\omega$ .

(b) Compute  $\text{curl}\mathbf{v}$ .

Vector fields which curl to zero are called *irrotational*.

(c) Compute  $\text{curl}F$ , where

$$F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6xz \\ 2y \\ 3x^2 \end{pmatrix}$$

Some intuition. If  $F$  is the velocity vector field of a fluid, then  $\text{curl}F$  at a point  $p \in \mathbb{R}^3$  points along the axis about which the fluid is rotating (locally near  $p$ ) and has length proportional to the speed of rotation at  $p$ .

(5) *Divergence*

(a) Compute the divergence of the vector field from 4(c) above.

(b) Compute the divergence of  $\nabla f$ , where  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  is smooth. (See also Ivan's Problem 5.)

(c) (Not to be handed in) Let  $F$  be the velocity vector field of a fluid in  $\mathbb{R}^3$ . Can you come up with an intuitive explanation of what  $\text{div}F$  measures? Hint: think about the relationship between  $\text{div}F$  at  $p \in \mathbb{R}^3$  and the limit as  $\epsilon \rightarrow 0$  of the net flow of a fluid out of cube of side  $2\epsilon$  centered at  $p$ .

Vector fields  $F$  such that  $\text{div}F$  is zero are called incompressible.

## 2 Ivan's problems

(1) *Volume forms and determinants*

Problem 4–3 on page 85 of Spivak. (Note that  $T$  is the inner product on the vector space  $V$ , page 83 also has this notation.)

(2) *Pushforward and Pullback*

Problem 4–13 part (a) on page 96 of Spivak.

(3) *Tangent vectors to curves and curves on spheres*

(a) Problem 4–14 on page 96 of Spivak.

(b) Problem 4–16 on page 96 of Spivak.

(4) *Vector fields and flowlines*

A curve

$$\begin{aligned}\gamma : \mathbb{R} &\rightarrow \mathbb{R}^n \\ t &\mapsto \gamma(t)\end{aligned}$$

is called a *flowline* of a vector field  $F : \mathbb{R}^n \rightarrow T\mathbb{R}^n$  if and only if the tangent vector of  $\gamma$  at  $t$  IS the value of the vector field at the point  $\gamma(t)$ . (In the notation of Problem 4–14 this means  $\gamma_*((e_1)_t) = F(\gamma(t))$ , where (by abusing notation) we regard  $F(\gamma(t))$  as an element of the tangent space  $T_{\gamma(t)}\mathbb{R}^n$ .) To gain an intuitive understanding of a flowline, think of the vector field  $F$  as the velocity field of a fluid. Then a flowline  $\gamma$  represents the path of a particle moving in the fluid.

(a) Sketch the vector field on  $\mathbb{R}^2$  given by  $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ xy \end{pmatrix}$  and sketch the flowlines through  $(1, 0)$ ,  $(-4, 0)$  and  $(2, 2)$ .

(b) For each of these flowlines, compute the function  $\gamma(t)$ .

(5) We now follow up on Alison's problem 5(b). If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  has continuous second-order partial derivatives, the *Laplacian* of  $f$  is defined to be  $\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}$ .

With  $f$  as above, we say that  $f$  is *harmonic* on the open set  $U \subseteq \mathbb{R}^n$  provided that  $\nabla^2 f(x) = 0$  for all  $x \in U$ .

(a) Find a (simple) condition on the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + k$  that makes  $f$  harmonic.

(b) Show that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{1}{\|x\|^{n-2}}$  is harmonic on  $U = \mathbb{R}^n \setminus \{0\}$ .

(c) Show that if  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  is harmonic, then  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $f(x, y) = g(e^x \cos y, e^x \sin y)$  is also harmonic.

### 3 Supplementary problems — optional.

(1) *Orientation and sign of determinant.*

(a) Evaluate the integral  $\int \int_R e^{(x+y)/(x-y)}$ , where  $R$  is the trapezoidal region with vertices  $(1, 0)$ ,  $(2, 0)$ ,  $(0, -2)$  and  $(0, -1)$ . (Hint:  $u = x + y$ ,  $v = x - y$ .) Look at the boundaries of the region in the  $xy$ -plane and the  $uv$ -plane. Walk around one of them clockwise, under the linear transformation what direction will you walk about the other?

(b) Another (simpler) example: Consider the  $xy$ -plane. Now consider the linear transformation that takes  $x$  to  $y$  and  $y$  to  $x$ . What is the determinant of this map? Look at the square bounded by vertices  $(0, 0)$ ,  $(0, 1)$ ,  $(1, 1)$  and  $(1, 0)$ . Walk clockwise around this square in the  $xy$ -plane. What happens in under the transformation?

(2) Are you confused about how tangent vectors of curves fits in with tangent lines of the image curve? (That is, how the new understanding of tangent vectors fits in with your previous knowledge.) Then take a look at Problem 4–15 on page 96 of Spivak. This should clarify things for you.

(3) Problem 4–17 and 4–18. These give a nice view of vector fields, div and grad.

(3) We will be doing a problem similar to 4–19 on HW 12.