Math 25b Homework 12

Due Friday 5th May 2006.
ABSOLUTELY NO EXTENSIONS ALLOWED ON THIS HOMEWORK!

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1   Alison's problems

(1) *Filling in details of proofs about d*
   (a) Problem 4-13 part (b) on page 96 of Spivak. (Note: you can't use Theorem 4–10 (2). This question asks you to fill in the missing part of the proof of that result.)
   (b) Fill in the detail missing in the proof of Theorem 4–10 (4). That is if ω is a 0-form and $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then $f^*(d\omega) = d(f^*\omega)$.

(2) *How does d relate to div, grad and curl?*
   (a) Show that the following diagram is commutative:

$$0 \xrightarrow{\quad} \text{0-forms} \xrightarrow{\ d\ } \text{1-forms} \xrightarrow{\ d\ } \text{2-forms} \xrightarrow{\ d\ } \text{3-forms} \xrightarrow{\quad} 0$$

$$\uparrow \qquad \uparrow R \qquad\qquad \uparrow S \qquad\qquad \uparrow T \qquad\qquad \uparrow U \qquad\qquad \uparrow$$

$$0 \xrightarrow{\quad} \text{functions} \xrightarrow{grad} \text{vector fields} \xrightarrow{curl} \text{vector fields} \xrightarrow{div} \text{functions} \xrightarrow{\quad} 0$$

(So this means you need to show that each square commutes. For example, $S \circ grad = d \circ R$.)
In this diagram all vector fields, differential forms, and functions are on $\mathbb{R}^3$ and

$$R(f) = f$$

$$S \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix} = F_1 dx + F_2 dy + F_3 dz$$

$$T \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix} = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

$$U(f) = f\, dx \wedge dy \wedge dz$$

(b) Deduce that
$$curl(grad(f)) = 0$$
for all functions $f : \mathbb{R}^3 \to \mathbb{R}$ and that
$$div(curl\mathbf{F}) = 0$$
for all vector fields $\mathbf{F}$ on $\mathbb{R}^3$.

(3) *A closed form that is not exact*
  Consider the 1-form defined on $A = \mathbb{R}^2 \smallsetminus \{(0,0)\}$:
$$\omega = -\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy.$$

  (a) Is ω closed? In other words, does $d\omega = 0$?
  (b) Let C_0 be the unit circle in $\mathbb{R}^2$ parameterized by the function $\gamma_0 : [0, 1] \to \mathbb{R}^2$, where $\gamma_0(t) = (\cos 2\pi t, \sin 2\pi t)$. Find $\int_{C_0} \omega$.
  (c) Is ω exact? In other words, does there exist a function $f : A \to \mathbb{R}$ such that $df = \omega$?
  (d) Is the restriction of ω to $B = \mathbb{R}^2 \smallsetminus \{(x, 0) : x \le 0\}$ exact?
  For more on closed and exact forms see Ivan's problems (3) and (4).

(4) *Line Integrals*
  (a) Compute the work done by a force field
$$\mathbf{F}(x, y, z) = \begin{pmatrix} y^2 \cos z \\ 2xy \cos z \\ -xy^2 \sin z \end{pmatrix}$$

moving a particle from $(2, 0, 0)$ to $(0, 0, 3)$ along the curve which is the intersection of the half ellipsoid
$$9x^2 + y^2 + 4z^2 = 36, \qquad z \ge 0$$

with the xz-plane.
(Hint: there is a fast way to do this question.)
  (b) Suppose that $\mathbf{F}$ is a velocity field of a fluid moving in $\mathbb{R}^2$ such that one of the flowlines of $\mathbf{F}$ is a closed curve. Can $\mathbf{F}$ be conservative?

2   Ivan's problems

(1) *Gauss' Law*
  The electric field $\mathbf{E}$ at a point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ not at the origin created by a point charge of charge Q at the origin is equal to
$$\frac{Q\epsilon x}{||x||}$$

2

where ϵ is a constant. (This is the electric force by a unit charge—a "test charge"—placed at x.)

(a) Write down the 2-form corresponding to the vector field **E**. Is it closed?

(b) Compute the flux of **E** outwards through a sphere of radius $r > 0$ centered at the origin. (You should do this directly—calculate an appropriate integral.)

(c) Why do (a) and (b) not contradict Stokes's Theorem?

(d) Explain why Stokes's Theorem implies *Gauss' Law*: the electric flux out of a smooth closed surface $S \subset \mathbb{R}^3$ is proportional to the charge enclosed by the surface.

(2) *Stokes's Theorem*

(a) Use Stokes's Theorem to evaluate

$$\int_C -y^2 dx + x dy + z^2 dz$$

where C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented anticlockwise when viewed from above.

(b) Let C be a smooth closed curve in $\mathbb{R}^2$, oriented anticlockwise. Show that

$$\int_C x dy = -\int_C y dx = \frac{1}{2} \int_C x dy - y dx = \text{the area bounded by } C.$$

Optional extra: What is the area of the polygon in $\mathbb{R}^2$ with vertices $(x_1, y_1), \ldots, (x_n, y_n)$?

(c) This part is optional. **Eureka!** Problem 5–36 on page 137. For this problem "three manifold with boundary" just means a bounded region in $\mathbb{R}^3$ bounded by a smooth surface S.

Homology

We now follow up on Alison's Problem (3). A sequence of vector spaces and linear maps

$$\cdots \xrightarrow{T_{i-1}} V_i \xrightarrow{T_i} V_{i+1} \xrightarrow{T_{i+1}} V_{i+2} \xrightarrow{T_{i+2}} \cdots$$

is called a *complex* if $T_{i+1} \circ T_i = 0$ for all i. Exact sequences are examples of complexes. The sequence

$$0 \longrightarrow \Omega^0_{dR}(A) \xrightarrow{d} \Omega^1_{dR}(A) \xrightarrow{d} \Omega^2_{dR}(A) \xrightarrow{d} \Omega^3_{dR}(A) \longrightarrow 0$$

for an open set A of $\mathbb{R}^3$ is another example, called the *de Rham complex of A*. Given a complex, we define the *ith homology group* of the complex to be the quotient vector space

$$H_i = \ker T_i / \mathrm{im} T_{i-1}.$$

Note that this makes perfect sense, as $\mathrm{im} T_{i-1}$ is a subspace of $\ker T_i$.

The ith homology group of the de Rham complex of A is called the *ith de Rham cohomology group* of A, and is writen

$$H^i_{dr}(A).$$

Roughly speaking, it measures how many closed i-forms on A are not exact.

(3) *Zeroth cohomology*

Suppose A is a connected open subset of $\mathbb{R}^n$. Compute $H^0_{dR}(A)$. What happens if A is not connected?

(4) *The Snake Lemma*

Suppose that we have a commutative diagram

$$
\begin{array}{ccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & A_{k+1} & \xrightarrow{f_{k+1}} & B_{k+1} & \xrightarrow{g_{k+1}} & C_{k+1} & \longrightarrow & 0 \\
\uparrow & & \uparrow{\scriptstyle d^A_k} & & \uparrow{\scriptstyle d^B_k} & & \uparrow{\scriptstyle d^C_k} & & \uparrow \\
0 & \longrightarrow & A_k & \xrightarrow{f_k} & B_k & \xrightarrow{g_k} & C_k & \longrightarrow & 0 \\
\uparrow & & \uparrow{\scriptstyle d^A_{k-1}} & & \uparrow{\scriptstyle d^B_{k-1}} & & \uparrow{\scriptstyle d^C_{k-1}} & & \uparrow \\
0 & \longrightarrow & A_{k-1} & \xrightarrow{f_{k-1}} & B_{k-1} & \xrightarrow{g_{k-1}} & C_{k-1} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\end{array}
$$

where the rows are exact and the columns are complexes. (So the diagram continues above and below what is shown.) Such an object is known as a *short exact sequence of complexes.*

(a) Show that the maps $f_* : H_k(A) \to H_k(B)$ where $[\omega] \mapsto [f_k(\omega)]$ and $g_* : H_k(B) \to H_k(C)$ where $[\eta] \mapsto [g_k(\eta)]$ are well defined.

Recall that the kth cohomology group is a quotient space $H_k(A) = \ker(d^A_k)/\mathrm{im}(d^A_{k-1})$. Here $\omega \in \ker(d^A_k)$ and $[\omega]$ is the equivalence class (or coset) $\omega + \mathrm{im}(d^A_{k-1})$. A similar definition holds for $[\eta]$.

(b) Construct a map $h_* : H_k(C) \to H_{k+1}(A)$ such that the following sequence is exact:

$$\dots \longrightarrow H_k(A) \longrightarrow H_k(B) \longrightarrow H_k(C) \longrightarrow H_{k+1}(A) \longrightarrow H_{k+1}(B) \longrightarrow \dots$$

In other words, *from a short exact sequence of complexes we can construct a long exact sequence of homology groups.* This will turn out to be very useful!

Having trouble getting started?

(i) Start with $\omega \in \ker(d^C_k)$ and try to construct an element $\nu \in \ker(d^A_{k+1})$. In the end you want $[\nu] \in H_{k+1}(A)$ so you really want the image of your map to be well defined up to an

4

ambiguity of the form $\nu \approx \nu + d_k^A(\phi)$. Now the map you want doesn't start at ω, but at $[\omega]$. So we need to check that if we change ω by an element of $\text{im}(d_{k-1}^C)$ then $[\nu] \in H_{k+1}(A)$ doesn't change. (I can give you more of a hint as to how to go from ω to ν if you need it.)
(ii) Now you've defined h_* don't forget to check that the sequence is exact!

3   References and suggestions.

(1) Make sure you complete all the details in the worksheet from class on Friday 28th April. Also please work hard to understand the cohomology questions on this assignment. You can expect more questions on cohomology on the final exam that will build on this material!

(2) Additional references to Spivak:
a. The geometry of physics: an introduction by Theodore Frankel.
b. Geometrical methods of mathematical physics by Bernard Schutz. (Note: I currently have Cabot's copy out.)
c. Differential Topology by Guilleman and Pollack. Chapter 4 of this book goes over everything we are doing on tensors, forms and intergration, but over manifolds rather than $\mathbb{R}^n$.
d. Introduction to Differentiable Manifolds and Riemannian Geometry by William Boothby. (This is the text I'll use in Math 134 next year. It is very complete. The downside for you at the moment is the notation is slightly different to what we've been using, so you'll have to "translate" everything.)

(3) Additional references on (de Rham) cohomology:
a. Algebraic Topology: a first course by William Fulton.
b. Differential forms in Algebraic Topology by Bott and Tu—a classic text!
c. Math 135 is currently using a text called: From calculus to cohomology; de Rham cohomology and characteristic classes by Ib Madsen and Jorgen Tornehave. I haven't looked at this book, but Professor Eftekhary has relayed the fact that it really does start at calculus and goes from there.