

Math 25b Homework 1

Due Tuesday 7th February 2006.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) This question looks at $n \times m$ matrices over \mathbb{R} . (In each case they are written with respect to the standard basis.)

(a) Find a basis for the kernel of the following two matrices (do not use technology):

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 4 \end{pmatrix}.$$

(b) Find a basis for the image of the following two matrices (do not use technology):

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 1 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{pmatrix}.$$

For the questions 2, 3 and 4 we assume $V = \mathbb{R}^n$ over \mathbb{R} with the standard inner product.

We define the set of all linear transformations (and their matrices with respect to the standard basis) from V to V to be:

$$M_n(\mathbb{R}) = \{A : V \rightarrow V \mid A \text{ is linear}\}.$$

(2) (a) Show that $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$.

(b) Show that $\det : M_n(\mathbb{R}) \rightarrow \mathbb{R}$ is continuous. (Hint: this part is important for the rest of the questions.)

We define the collection of invertible linear transformations, called the *general linear group* to be

$$GL_n(\mathbb{R}) = \{A : V \rightarrow V \mid A \text{ is invertible}\}.$$

We further define two special subgroups of the general linear group, called the *special linear group* and the *orthogonal group*, respectively, as follows:

$$SL_n(\mathbb{R}) = \{A : V \rightarrow V \mid \det A = 1\},$$

$$O_n(\mathbb{R}) = \{A : V \rightarrow V \mid A^t A = I\}.$$

(Here A^t is the transpose of the matrix A .)

(3) (a) Show that $GL_n(\mathbb{R})$ is open as a subset of $M_n(\mathbb{R})$.

(b) Show that $SL_n(\mathbb{R})$ is closed (for $n \geq 1$), but not compact (for $n \geq 2$).

(4) (a) Show that a real $n \times n$ matrix has orthonormal columns if and only if $A^t A = I$. (So the each column of the matrix can be considered as a vector of length 1 and the columns are pairwise orthogonal.)

(b) Show that $O_n(\mathbb{R})$ is compact in $M_n(\mathbb{R})$. (Hint: (a) may be useful in showing $O_n(\mathbb{R})$ bounded.)

2 Ivan's problems

(1) Decide which of the matrices A are diagonalizable. If possible, find a diagonal matrix D and an invertible matrix S such that $S^{-1}AS = D$. (Do not use technology.)

$$A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \quad A = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We have the following metric spaces. (Check that they are indeed metric spaces but don't hand in!)

(a) \mathbb{R}^n , $d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|$

(b) \mathbb{R}^n , $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$

(c) \mathbb{R}^n , $d_\infty(x, y) = \max\{|x_1 - y_1|, \dots, |x_n - y_n|\}$

(d) $C([0, 1])$, $d_1(f, g) = \int_0^1 |f(x) - g(x)| dx$

(e) $C([0, 1])$, $d_\infty(f, g) = \max_x |f(x) - g(x)|$

(Here $C([0, 1])$ is the set of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$. We will prove later that integrals and maxima always exist for such functions.)

(2) For the metrics d_1, d_2, d_∞ on \mathbb{R}^2 , draw the open unit ball around 0. (This is denoted $B_1(0) = \{x \in \mathbb{R}^2 \mid d(x, 0) < 1\}$.)

(3) Two metrics d_1 and d_2 on a set S are said to be *equivalent* if for any point $p \in S$, every open ball $B_R^1(p)$ with respect to the first metric contains an open ball $B_r^2(p)$ with respect to the second metric, and vice versa, every open ball $B_r^2(p)$ with respect to the second metric contains an open ball $B_R^1(p)$ with respect to the first metric.

(a) Prove that if two metrics d_1 and d_2 are equivalent on S , then a sequence $\{a_i\}$ in S converges with respect to d_1 if and only if it converges with respect to d_2 .

(b) Now prove that on \mathbb{R}^n , the metrics d_1, d_2, d_∞ are all equivalent.

(c) Prove that the metrics d_1 and d_∞ on $C([0, 1])$ are not equivalent by finding a sequence that converges with respect to d_1 but does not converge with respect to d_∞ .