Math 25b Homework 2

Due Wednesday 15th February 2006.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1   Alison's problems

(1) (a) Let $V = \mathbb{R}^n$ and recall from HW 1 that $M_n(\mathbb{R}) = \{A : V \to V \mid A \text{ is linear}\}$ and $GL_n(\mathbb{R}) = \{A : V \to V \mid A \text{ is invertible}\}$. Show that the closure of $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ is all of $M_n(\mathbb{R})$.

(b) Let X be the collection of diagonalizable matrices in $M_2(\mathbb{R})$. Is X open, closed or neither? Explain.

(2) Problem 2 on page 98 of Rudin.

(3) Problem 4 on page 98 of Rudin. (Please also read but not submit problems 5 and 13.)

(4) Problem 6 on page 99 of Rudin.

(5) Problem 14 on page 100 of Rudin.

2   Ivan's problems

(1) Problem 8 on page 99 of Rudin.

(2) Problem 17 on page 100 of Rudin.

(3) Problem 18 on page 100 of Rudin.

(4) Let (X, d) be a metric space. A function $f : X \to X$ is called a *contraction* if and only if there exists $\alpha \in [0, 1)$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \quad \text{for all } x, y \in X.$$

A point $p \in X$ is called a *fixed point* if $f(p) = p$.

(a) Prove that every contraction of a complete metric space has a unique fixed point. *(This is known as the Contraction Mapping Theorem.)*

(b) Can you find an example of a function $g : X \to X$ satisfying

$$(\star) \qquad d(g(x), g(y)) < d(x, y) \quad \text{for all } x, y \in X \text{ with } x \neq y$$

with no fixed point? Can such a function have several different fixed points?

(c) If X is compact and $g : X \to X$ satisfies $(\star)$, show that g has a unique fixed point.

(d) Give an example of a compact metric space X and $g : X \to X$ satisfying $(\star)$ such that g is not a contraction.

3 The problems I didn't assign—good practice!

(1) Problem 7 on page 99 of Rudin.
(2) Problem 9 on page 99 of Rudin.
(3) Problem 12 on page 99 of Rudin.
(4) Problem 23 on page 101 of Rudin.
(5) Problem 26 on page 102 of Rudin.