Math 25b Homework 4

Due Wednesday 1st March 2006.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

Recall that a *normed space* consists of a vector space V together with a function

$$||\cdot|| : V \to \mathbb{R}$$

that assigns a vector $v \in V$ to its *norm* $||v||$, satisfying the following properties for all $v, w \in V$, $\alpha \in \mathbb{R}$:

- $||v|| \geq 0$,
- $||v|| = 0$ if and only if $v = 0$,
- $||\alpha v|| = |\alpha| ||v||$,
- $||v + w| \leq ||v|| + ||w||$.

(1) (a) Prove that if $(V, ||\cdot||)$ is a normed space, then $d(v, w) = ||v - w||$ is a metric on V.

(b) Consider the five metrics defined in HW 1. Show that each metric comes from a norm. Convince yourself that these really are norms. Just submit one of these metric/norms for grading and think through the rest.

Let $||\cdot||_\alpha$ and $||\cdot||_\beta$ be two norms on a vector space V. We say that $||\cdot||_\alpha \leq ||\cdot||_\beta$ if there exists a constant $C > 0$ such that $||v||_\alpha \leq C||v||_\beta$ for any $v \in V$. The two norms are said to be *equivalent* if $||\cdot||_\alpha \leq ||\cdot||_\beta$ and $||\cdot||_\beta \leq ||\cdot||_\alpha$.

(2) The goal of this question is to prove that all norms on $\mathbb{R}^n$ are equivalend by showing that an arbitrary norm $||\cdot||_\alpha$ on $\mathbb{R}^n$ is equivalent to the norm $||\cdot||_1$ defined in problem (1b) (the "sum of absolute values" norm).

(a) Let $M = \max_i ||\vec{e_i}||_\alpha$. Using the triangle inequality prove that for any $v \in \mathbb{R}^n$, $||v||_\alpha \leq M||v||_1$.

(b) Prove that the function $||\cdot||_\alpha : \mathbb{R}^n \to \mathbb{R}$ is continuous.

(c) Let $S = \{v \in \mathbb{R}^n : ||v||_1 = 1\}$ be the unit sphere with respect to the norm $|| \cdot ||_1$ (so it is actually the boundary of a "diamond"). We consider the norm $|| \cdot ||_\alpha$ restricted to S:

$$|| \cdot ||_\alpha : S \to \mathbb{R}.$$

Let m be the minimum value of this function. (Why must this be achieved?) Prove that $m \neq 0$ and $||v||_1 \leq \frac{1}{m}||v||_\alpha$ for any $v \in \mathbb{R}^n$.

(3) Problem 2 on page 138 of Rudin

(4) Problem 4 on page 138 of Rudin.

2 Ivan's problems

(1) Problem 5 on page 138 of Rudin.

(2) Problem 6 on page 138 of Rudin.

(3) Problem 8 on page 138—9 of Rudin. (Now in a parallel universe I also set Problem 9 on page 139 of Rudin. However, this is a long problem to do and would make this assignment a bit of a killer. So please read it through, think about it and appreciate how useful problems 7 and 8 on page 138 of Rudin are.)

(4) Problem 10 parts a—c on page 139 of Rudin.

3 The problems I didn't assign—good practice!

(1) Problem 3 on page 138 of Rudin.
(2) Problems 7 and 9 on page 138—9 of Rudin.
(3) Problem 11 on page 140 of Rudin.
(4) Problem 16 on page 141 of Rudin.