Math 25b Homework 6

Due Wednesday 15th March 2006.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) Problem 7 on page 166 of Rudin.

(2) Problem 11 on page 167 of Rudin.

(3) Problem 13 on page 167 of Rudin.

(4) Problem 15 on page 168 of Rudin.

2 Ivan's problems

(1) Problem 16 on page 168 of Rudin.

(2) Problem 19 on page 168 of Rudin.

(3) Problem 24 on page 170 of Rudin.

(4) Given the differential equation $zf''(z)+f'(z) = zf(z) = 0$, you are to find a solution $f(z)$ (not the zero function) of the differential equation which is the sum of a power series with nonzero radius of convergence R: $f(z) = \sum_{k=0}^{\infty} a_k z^k$. We shall also assume that $f(0) = 1$. Show that

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}(k!)^2} z^{2k} = 1 - \frac{z^2}{4} + \frac{z^4}{64} - \frac{z^6}{2^6(3!)^2} + \dots$$

and that the radius of convergence is ∞.

(Hint: Substitute, rearrange and use induction to get the expression for $f(z)$. For the radius of convergence you may use/cite the fact that $\sum_{k=0}^{\infty} |z|^{2k}/k!$ converges (to $e^{|z|^2}$) for any $z \in \mathbb{C}$.)

Remarks:

1) The function f you've found is usually denoted $J_0(z)$ and is known as Bessel's function of order 0.

2) If you think about it, you've found *all* power series solutions of the differential equation satisfying $f(0) = 1$. (Power series expansions are unique.) Interestingly enough, the differential equation has another solution which is *not* representable as a power series, but we won't get into this here.

3 The problems I didn't assign—good practice!

(1) Problem 25 on page 170 of Rudin.

(2) Show that $Card(\mathbb{R}^{\mathbb{N}}) \neq Card(\mathbb{R}^{\mathbb{R}})$. Deduce that there exists a function $f : [0, 1] \to \mathbb{R}$ which is not the pointwise limit of continuous fuctions. This is a very hard problem that I learned from Tom Coates. You'll need to use (implicitly) some of the things about countability etc that we learned in Fall semester.