Math 25b Homework 8

Due Wednesday 5th April 2006.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) *Directional derivatives and grad.*

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. For $a \in \mathbb{R}^n$, set

$$\nabla f(a) = \begin{pmatrix} D_1 f(a) \\ \vdots \\ D_n f(a) \end{pmatrix}.$$

The vector $\nabla f(a)$ is called the *gradient of* f *at* a or *grad* f *at* a; the function

$$\nabla f : \mathbb{R}^n \to \mathbb{R}^n$$

is often called "grad$f$".

(a) Look at the definition of directional derivative $D_x f$ in problem 2-29 on page 33 of Spivak. Show that

$$D_x f(a) = \langle \nabla f(a), x \rangle.$$

(Here, $\langle , \rangle$ denotes the standard inner product. You could also write this using dot product $D_x f(a) = \nabla f(a) \cdot x$.)

(b) Show that $\nabla f(a)$ vanishes whenever a is a local maximum or minimum of f.

(c) Show that $\nabla f(a)$ points in the direction of steepest increase of f at a. What does the length $|\nabla f(a)|$ represent?

(2) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(a) Show that f is differentiable at 0 and that $f'(0) = \frac{1}{2}$.

(b) Show that there is no open interval containing 0 on which f is one-to-one.

Why doesn't this contradict the Inverse Function Theorem?

2 Ivan's problems

(1) *Level surfaces*

(a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose that $f(a, b, c) = k$, and that

$$\frac{\partial f}{\partial z}(a, b, c) \neq 0.$$

Show that there is an open set $U \subset \mathbb{R}^2$ containing (a, b) and a differentiable function $g : U \to \mathbb{R}$ such that

$$f(a, b, g(a, b)) = k.$$

Explain why the points in the set

$$\{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = k\}$$

which are sufficiently close to (a, b, c) form a smooth surface in $\mathbb{R}^3$.

(b) Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and that $k \in \mathbb{R}$ is such that the partial derivatives

$$\frac{\partial f}{\partial x} \qquad \frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial z}$$

do not simultaneously vanish anywhere on the set

$$f^{-1}(k) = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = k\}.$$

Explain why the set $f^{-1}(k)$ forms a smooth surface in R^3. $f^{-1}(k)$ is called the level surface of f at level k. *You need to show that near each point $(a, b, c) \in f^{-1}(k)$, the set $f^{-1}(k)$ looks like a smooth surface in $\mathbb{R}^3$.*

(c) Show that if $(a, b, c) \in f^{-1}(k)$, then $\nabla f(a, b, c)$ is perpendicular[1] to the surface $f^{-1}(k)$ at the point (a, b, c).
So "gradf points perpendicularly out of level surfaces of f".

(d) Draw a picture containing some level surfaces $f^{-1}(k)$ and some values of ∇f for

$$f(x, y, z) = x^2 + y^2 + z^2.$$

(e) Can you see how your answers to Alison's 1(c) and Ivan's 1(c) fit the picture you have drawn? (No need to hand this one in.)

[1] So what does it mean to be perpendicular to a surface, when the surface is not a plane? Draw a picture and have a think about the following definition. We say that a vector $v \in \mathbb{R}^3$ is perpendicular to the surface $S \subset \mathbb{R}^3$ at the point $x \in S$ if and only if for each smooth curve

$$\begin{aligned} \gamma : &\mathbb{R} \to \mathbb{R}^3 \\ &t \mapsto \gamma(t) \end{aligned}$$

such that $\gamma(t) \in S$ for all t and $\gamma(0) = x$ the vector $\gamma'(0)$ is perpendicular to v.

(2) *Injectivity and local injectivity*

Problem 2-38 on pages 39-40 of Spivak. Show in addition that for the function f in part (b), we can find a neighborhood U of any point $(a, b) \in \mathbb{R}^2$ such that $f|_U$ is injective. So f is *locally injective* but not injective.