Math 25b Homework 9

Due Wednesday 12th April 2006.

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) *Second derivative test*

Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that it has continuous third partial derivatives at each point of an open ball B centered at $x \in \mathbb{R}^n$.

(a) Show that for $h = (h^1, \ldots, h^n) \in \mathbb{R}^n$ sufficiently small there is some point y on the straight line between x and $x + h$ such that

$$f(x+h) = f(x) + \sum_{i=1}^{n} D_i f(x) h^i + \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} D_{ij} f(x) h^i h^j + \frac{1}{3!} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} D_{ijk} f(y) h^i h^j h^k$$

(Hint: You could apply Taylor's Theorem to the function $g : \mathbb{R} \to \mathbb{R}$ such that $g(t) = f(x + th)$.)

(b) Let M be the matrix with (i, j)th entry $D_{ij} f(x)$ and observe that

$$\frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} D_{ij} f(x) h^i h^j = h^T M h.$$

Show that M is diagonalizable. Let $D = \det M$ and $T = \operatorname{tr} M$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and $(x, y) \in \mathbb{R}^2$ is a critical point. (So $D_1 f(x, y) = 0 = D_2 f(x, y)$.) Show that

- if $D > 0$ and $T > 0$, x is a local minimum;
- if $D > 0$ and $T < 0$, x is a local maximum;
- if $D < 0$, then x is neither a min nor a max.

Of course of $D = 0$ then we'd need to use some other method to work out what is going on. (If the function is smooth enough we could look at the third derivative.)

(Hint: you can actually do a large part of (b) in the n-dimensional setting. You might also find it helpful to look at the Rayleigh-Ritz Theorem.)

(c) Find classify the critical points of the function $f(x, y) = x^4 + y^4 - 4xy + 1$.

(2) *Tangent Planes*

(a) Find the equation of the tangent plane to $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

(b) Find the equation of the tangent plane to $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$.

(c) A surface is given by $f : \mathbb{R}^2 \to \mathbb{R}^3$ with $f(u, v) = (u^3, u \sin v, u \cos v)$. (Another way of saying this is that the surface is parametrized by u and v and a point (x, y, z) on the surface is found by parametric equations $x = u^3$, $y = u \sin v$, $z = u \cos v$.) Find the equation of the tangent plane to the surface at the point $(8, 2, 0)$. Can you visualize what this surface looks like?

(Hint: in each part you'll have to find the normal vector to the tangent plane. This might need some thought.)

Remark: Surfaces of revolution can be thought of as parametric surfaces. If $y = f(x)$ is rotated about the x-axis (where $f(x) \geq 0$) Let θ be the angle of rotation (and $0 \leq \theta \leq 2\pi$) so that a point (x, y, z) on the surface of revolution is found by the parametric equations $x = x$, $y = f(x) \cos \theta$, $z = f(x) \sin \theta$.

(3) *Minimizing functions*

In this question I would like you to solve the following problem in three different ways: find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

(a) Method 1: Using the information given, find an equation of *two* variables that (when minimized) will solve the problem. (So once you've got the equation, find critical values and check for minima etc.)

(b) Method 2: Use Lagrange multipliers.

(c) Method 3: Solve this using vectors and the lengths of vectors. Hint: one vector you will want to think about is the normal vector to the plane. The other hint is to work out how to calculate the projection of vector $\vec{a}$ onto vector $\vec{b}$.

I can give more hints for any of these methods — just ask.

Remark: Now, can you find the distance between two parallel planes? Can you find the distance between two skew lines (lines that do not intersect and are not parallel)? Just think this through—you don't have to submit this.

Remark: There are many ways to find a solution to a problem. Sometimes one method will work better for than another. Can you think about what might be the strengths and weaknesses of Method 1 and Method 2? (You don't have to submit this.)

(4) *Measure zero*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the graph of f, which is a subset of $\mathbb{R}^2$, has measure zero. Now, think through how you might prove an analogous result for continuous functions $f : \mathbb{R}^n \to \mathbb{R}^n$ (you don't have to submit this however).

2 Ivan's problems

(1) *Lagrange Multipliers*

Let f and g be smooth functions from $\mathbb{R}^3 \to \mathbb{R}$. In this question we will develop a method for solving the following *constrained maximization problem*: find the maximum value $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. (That is only worry about the value of f for points (x, y, z) such that $g(x, y, z) = k$.) We also assume that:

• ∇g is not zero anywhere on $g^{-1}(k)$. (So you know from HW8 that the level set $g^{-1}(k)$ is a smooth surface);

• all level sets of $f^{-1}(l)$ are smooth surfaces[1].

Let M be the maximum value of f on the surface $g^{-1}(k)$ and suppose this maximum occurs at $(a, b, c) \in g^{-1}(k)$. We want to find M.

(a) Show that the level surfaces $f^{-1}(M)$ and $g^{-1}(k)$ are tangent at (a, b, c) **and** that (a, b, c) is a solution to the following system of equations

$$(*) \qquad \begin{cases} \nabla f = \lambda \nabla g(x, y, z) \\ g(x, y, z) = k \end{cases}$$

(**Warning:** please be careful about what you mean by perpendicular, parallel and tangent. Take a look at HW 8 again for the definition of what it means for a vector to be perpendicular to a surface.)

(b) Must any solution to (*) give either the maximum or minimum values of $f(x, y, z)$ on the set $g^{-1}(k)$?

(c) Use the technique of this question to find the minimum surface area of a rectangular box which has volume equal to 8 units.

(2) *Content zero subsets do not affect integrals*

Suppose that $A \subset \mathbb{R}^n$, $B \subset A$ such that $A \setminus B$ has content zero. Suppose that $f : A \to \mathbb{R}$ is a bounded function and that $g : B \to \mathbb{R}$ is the restriction of f to B. Show (i) that $\int_A f$ exists if and only if $\int_B g$ exists and (ii) that if the integrals exist then they are equal.

(3) *Some integrals*

(a) Prove that if $f(x, y)$ is a continuous function and R is the region

$$R = \{(x, y) \in \mathbb{R}^2 \,|\, a \le x \le b, g_1(x) \le y \le g_2(x)\},$$

where $g_1(x) \le g_2(x)$ for all $x \in [a, b]$ and the functions g_1 and g_2 are continuous, then

$$\int_R f(x, y)\, dxdy = \int_{x=a}^{x=b} \left(\int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy \right) dx.$$

[1]Note we can relax this assumption a little to "all level sets of f which meet a neighborhood of the surface $g^{-1}(k)$ are smooth surfaces in a neighborhood of $g^{-1}(k)$".

Such regions are "swept out by vertical lines". Can you state a similar result for regions "swept out by horizontal lines"?

(b) Sketch the region $S = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$ and evaluate

$$\int_S e^{x+y}\, dxdy.$$

(c) Sketch the solid bounded by the surface $z = x^2 - y^2$, the xy-plane and the planes $x = 1$ and $x = 3$ and compute its volume.

(4) *The Five Lemma*

Suppose that each row of the diagram

$$\begin{array}{ccccccccc}
V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & V_4 & \longrightarrow & V_5 \\
\big\downarrow {\scriptstyle T_1} & & \big\downarrow {\scriptstyle T_2} & & \big\downarrow {\scriptstyle T_3} & & \big\downarrow {\scriptstyle T_3} & & \big\downarrow {\scriptstyle T_3} \\
W_1 & \longrightarrow & W_2 & \longrightarrow & W_3 & \longrightarrow & W_4 & \longrightarrow & W_5
\end{array}$$

is an exact sequence of linear maps between vector spaces, and that $T_1, \ldots, T_5$ are linear maps such that T_1, T_2, T_4, and T_5 are isomorphisms. Show that T_3 is also an isomorphism. (We'll use this lemma later on in HW exercises.)

3 Supplementary problems — optional.

(1) How and under what conditions can we generalize the method of Lagrange multipliers to solve the following problem: find the extreme values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = l$?

(2) There is more on Lagrange multipliers in Spivak, namely Problems 5-16 and 5-17 on page 122.

(3) Need more practice with integrals or any of the multivariable material? Any multivariable calculus textbook will have plenty of examples. One that is in use (and on reserve in Cabot library) is Stewart's Multivariable Calculus textbook.