Math 25b Midterm Exam
Practice Questions

Last updated 8th March 2006

(1) Go over all your old homework assignments!

(2) Make sure you understand the main results from class and how to use them. Also make sure you understand the main techniques used in the proofs. There are a number of ideas that just keep coming up. (Looking at HW's also helps with both of these things.)

(3) I've included some practice questions below (they might be similar to questions on the exam). These are by no means exhaustive and are designed to help your study.

(4) Are the following TRUE or FALSE? (Make sure you can justify your answers by proof or counterexample if you need to.)

(a) If $f : [0, 1] \to \mathbb{R}$ is continuous at every irrational number, then f is Riemann integrable.

(b) If f has a power series centered about $x = 0$ whose radius of convergence is 1, then f is C^∞ on $(-1, 1)$.

(c) If X is a complete metric space and $f : X \to X$ is a contraction, then f has a unique fixed point.

(d) If a_n is a sequence of real numbers and $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$.

(e) If a_n is a bounded sequence of real numbers, then the power series $\sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$ converges to a continuous function on $\mathbb{R}$.

(f) If $f_n : [0, 1] \to \mathbb{R}$ is a sequence of differentiable function and f_n converges to f uniformly, then the sequence of derivatives f_n' converges to f'.

(5) Make sure you understand all the definitions we've been talking about and have examples on hand which distinguish between them. (For example: continuous vs uniformly continuous, sequences (or series) of functions converging pointwise vs converging uniformly, pointwise bounded vs uniformly bounded, equicontinuity, etc, etc.)

(6) Suppose $f : [0,1] \to \mathbb{R}$ is bounded and integrable and let

$$g(y) = \int_0^y f(x)\,dx \quad y \in [0,1].$$

Show that g is continuous. Is g always differentiable?

(7) Suppose $f : \mathbb{Q} \to \mathbb{R}$ is continuous. Is there necessarily a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = f(x)\ \forall x \in \mathbb{Q}$?

(8) (a) Suppose that K is a compact subset of X. Prove that if f is continuous on K, then f is uniformly continuous on K.

(b) Give an example to show that the conclusion in (a) is false if K is not compact.

(9) A function $f : (a,b) \to \mathbb{R}$ is said to be *convex* if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

whenever $a < x, y < b$, $0 < \lambda < 1$.

(a) Prove that every convex function is continuous.

(b) Prove that every increasing convex function of a convex function is convex.

(c) If f is convex in (a,b) and if $a < s < t < u < b$, show that

$$\frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}.$$

(10) Let $\alpha : [a,b] \to \mathbb{R}$ be monotonically increasing. Suppose $f : [a,b] \to \mathbb{R}$ is a continuous function. Show that f is Riemann-Stieltjes integrable with respect to α.

(11) Let $g_n : [a,b] \to \mathbb{R}$ $(n = 1, 2, \dots)$ be a sequence of functions such that $g_{n+1}(x) \leq g_n(x)$ for all $x \in [a,b]$ and all $n \in \mathbb{N}$. Suppose that $g_n \to 0$ uniformly on $[a,b]$. Prove that $\sum_{n=0}^{\infty} (-1)^n g_n(x)$ converges uniformly on $[a,b]$.