Math 25a Homework 9 Solutions

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1 Ivan's problems

(1) *Exact Sequences*
A sequence of linear maps between vector spaces

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-2}} V_{n-1} \xrightarrow{T_{n-1}} V_n$$

is called *exact* if and only if

$$im(T_i) = ker(T_{i+1}) \quad \text{for} \quad i = 1, 2, \ldots, n-2$$

For the rest of this question we write the zero vector space $\{0\}$ as 0.

(a) Show that

$$0 \longrightarrow V \xrightarrow{T} W \longrightarrow 0$$

is exact if and only if T is an isomorphism.

(b) Under what circumstances are

$$V \xrightarrow{T} W \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow V \xrightarrow{T} W$$

exact?

(c) Show that if

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n \longrightarrow 0$$

is exact, then

$$\sum_{i=1}^{n} (-1)^i \dim V_i = 0.$$

Solution. Based on Nike Sun's pset work

(a) Take $S \in L(0,V)$ and $U \in L(W,0)$ such that

$$0 \xrightarrow{S} V \xrightarrow{T} W \xrightarrow{U} 0.$$

This sequence is exact iff $rangeS = 0 = nullT$ iff T is injective and $rangeT = nullU = W$ (as the target space is 0) iff T is injective and surjective iff T is an isomorphism.

(b)$V \xrightarrow{T} W \xrightarrow{U} 0$ is exact iff $rangeT = nullU = W$ iff T is surjective. $0 \xrightarrow{S} V \xrightarrow{T} W$ is exact iff $rangeS = 0 = nullT$ iff T is injective.

(c)If $0 \xrightarrow{S} V_1 \xrightarrow{T_1} V_2 \rightarrow \ldots \rightarrow V_{n-1} \xrightarrow{T}_{n-1} V_n \xrightarrow{U} 0$ is exact, then $dimrangeS = dimnullT_1 = dimV_1 - dimrangeT_1 = dimV_1 - dimnullT_2 = dimV_1 - dimV_2 + dimrangeT_2 = \ldots = \sum_{i=1}^{n}(-1)^{i-1}dimV_i + (-1)^n dimrangeU = \sum_{i=1}^{n}(-1)^{i-1}dimV_i = 0$ since $dimrangeS = 0$ for all $S \in L(0,V_1)$. Therefore $\sum_{i=1}^{n}(-1)^{i}dimV_i = 0$.

□

(2) *Inverting matrices using elementary row and column operations*
An elementary row operation on a matrix is one of the following:

1. interchanging two rows
2. adding a multiple of one row to a different row
3. multiplying a row by a non-zero scalar.

Elementary column operations are the same but with "row" replaced with "column".

(a) Show that if matrices A and B are related by an elementary row operation then there exists an invertible matrix E such that $EA = B$. Show that if matrices A and B are related by an elementary column operation then there exists an invertible matrix E' such that $AE' = B$

(b) The *rank* of an $m \times n$ matrix A is the dimension of $\mathbb{R}^m$ spanned by the columns of A. Show that the rank of A is equal to the rank of the linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ given by $v \mapsto Av$. Show that the rank of a matrix is left unchanged by elementary row and column operations.

(c) Find out what the "reduced row-echelon form" of a matrix is. Convince yourself that one can always put a matrix into reduced row-echelon form using a sequence of elementary row operations. Do a couple of small examples. (Note: no need to submit the small examples for grading.)

(d) For the remainder of the question, assume that A is an invertible $n \times n$ matrix, or in other words an $n \times n$ matrix of rank n. Deduce that one can find a sequence $E_1, \ldots E_N$ of matrices such that

$$E_N E_{N-1} \ldots E_1 A = I$$

where I is the $n \times n$ identity matrix and E_i are matrices of the type you constructed in part (a).

(e) Show that if one makes a $2n \times n$ matrix by writing A next to the identity matrix I:

$$B = \begin{pmatrix} A & I \end{pmatrix}$$

and then performs elementary row operation on B until A becomes the identity matrix, then the resulting matrix B' is

$$B' = \begin{pmatrix} I & A^{-1} \end{pmatrix}$$

(f) Compute the inverse of the following matrix.

$$\begin{pmatrix} 1 & 6 & 5 \\ 0 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

(g) What happens if you try to use this method to compute the inverse of a non-invertible matrix A?

Solution.

1. It suffices to check that each of the elementary row operations may be encoded as left multiplication by an invertible matrix. Switching rows i and j corresponds to left-multiplication by $I_n + e_{ij} + e_{ji} - e_{ii} - e_{jj}$ [we let e_{ij} denote the matrix of all zeroes, with a single one at row i, column j – equivalently, the matrix corresponding to the linear map induced by $e_i \otimes e_j^\star$]; this is invertible, for it is its own inverse. Multiplying row i by $\alpha \neq 0$ corresponds to left-multiplication by $I_n + (\alpha - 1)e_{ii}$; this is invertible for its inverse is the corresponding matrix with α^{-1} in place of α. Adding row i to row j corresponds to left-multiplication by $I_n + e_{ji}$, which is invertible for it is upper or lower triangular with ones on the diagonal (so determinant one) [alternatively, because it has inverse $I_n - e_{ji}$ – subtracting row i from row j].

   The similar claim for column operations follows as column operations correspond to row operations on the transpose. So, writing A^T for the transpose of A, etc., by the above we may write $EA^T = B^T$ with E invertible; taking the transpose of both sides we get $AE^T = B$, with E^T invertible [by say, determinant calculations or an exact sequence and dual-map argument].

2. Let T be the specified linear transformation. Note that $\operatorname{im} T$ is precisely the space spanned by the column vectors of A [for letting e_i be the standard basis for $\mathbb{R}^n$, we get $T\left(\sum_i a_i e_i\right) = \sum_i a_i T(e_i)$, and $T(e_i)$ is precisely a column of A]. So, $\operatorname{rk} T = \dim \operatorname{im} T = \operatorname{rk} A$.

   By (a), we see that elementary row (resp. column) operations correspond to left (resp. right) multiplication by invertible matrices. Note that for $T : U \to V$ and $S : V \to V$ injective, we have by rank-nullity that $\dim \operatorname{im} ST = \dim \operatorname{im} T - \dim \ker S|_{\operatorname{im} T} = \dim \operatorname{im} T$. For $T : U \to V$ and $S : U \to U$ surjective, we have $\operatorname{im} T = \operatorname{im} TS$, so $\dim \operatorname{im} T = \dim \operatorname{im} TS$. So, row and column operations do not change the rank.

3. A matrix is in "reduced row-echelon form" if it satisfies the following properties:

   - The first non-zero element of a row occurs to the right of the first non-zero entry of the previous row.
   - The first non-zero element in any row is a 1.
   - THe first non-zero element in any row is the only non-zero value in its column.

   We may put a matrix into this form using "Gaussian elimination." At each step, we may use elementary row operations to shift the row with the left-most non-zero entry to the top, divide it to make this leading entry 1, and then subtract multiples of it from the other rows to zero out their entries in that column.

4. Each of the transformations involved in Gaussian elimination, as described above, is an elementary row operation (switching two rows, multiplying a row by a constant multiple, adding a constant multiple of one row to another). [Well, the last isn't really an elementary operation, but it may be accomlished by multiplying the row in question by the desired constant, adding it to the other row, and then multiplying it by the reciprocal of the constant.] So, our desired result follows from (a) once we show that the reduced row-echelon form of an invertible matrix is the identity matrix.

Note that the rank is maintained by elementary row operations, so the reduced row-echelon form of a matrix must have the same rank as the original matrix. If an $n\times$ matrix is invertible, then it has rank n. So, the reduced row-echelon form must have n ones, each to the right of the preceeding one, and with each the only non-zero one in its column. This forces the reduced row-echelong form to be the identity matrix.

5. We note that we may encode the row operations we must do to take A to I by left-multiplication by matrices $(E_1, \dots, E_N)$ as in the previous part. Then fact that inverses are unique, we have that $E_N \cdots E_1 = A^{-1}$. Now, the results of performing these row operations on B will be: $E_N \cdots E_1 \begin{pmatrix} A & I \end{pmatrix} = \begin{pmatrix} E_N \cdots E_1 A & E_N \cdots E_1 \end{pmatrix} = \begin{pmatrix} I & A^{-1} \end{pmatrix}$.

6. The inverse is
$$\frac{1}{7}\begin{pmatrix} 1 & 17 & 6 \\ 1 & 3 & -1 \\ 0 & -7 & 0 \end{pmatrix}$$
.

7. If you apply this method to compute the inverse of a non-invertible matrix, then Gaussian elimination will give you a row in the reduced row-echelon form that is all zeros. In particular, the product with your "almost inverse" matrix will end up having rows which are all zero and will not be the inverse matrix (as we'd expect, for the rank of the product can't be n). $\square$