Math 25a Homework 11 Solutions

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1 Alison's problems

(1) (a) Problem 6 on page 94 of Axler.
(b) Problem 8 on page 94 of Axler.

Solution. (a) Suppose λ is an eigenvalue. Then there exists a nonzero vector (z_1, z_2, z_3) such that $T(z_1, z_2, z_3) = \lambda(z_1, z_2, z_3)$. Comparing each component, we get the system of equations

$$2z_1 = \lambda z_2 \quad 0 = \lambda z_2 \quad 5z_3 = \lambda z_3$$

First suppose λ is nonzero. Then the second equation implies that $z_2 = 0$, and the first that $z_1 = 0$ as well. Because our eigenvector is nonzero, we must have z_3 nonzero, so that $\lambda = 5$, and $(z_1, z_2, z_3) = (0, 0, z_3)$ is the only solution.

Now suppose $\lambda = 0$. Then $(2z_2, 0, 5z_3) = (0, 0, 0)$, so z_2, z_3 must be 0, and z_1 can be anything. So the eigenvectors for $\lambda = 0$ are $(z_1, z_2, z_3) = (z_1, 0, 0)$.

(b) Suppose $z = (z_1, z_2, z_3, \ldots,)$ is an eigenvector of T with eigenvalue λ. Then $Tz = \lambda z$, which, taken componentwise, gives us the infinite system on equations $z_2 = \lambda z_1, z_3 = \lambda z_2, z_4 = \lambda z_3, \ldots$. We can solve for all the other coordinates in terms of z_1: $z_2 = \lambda z_1$, $z_3 = \lambda z_2 = \lambda^2 z_1$, $z_4 = \lambda z_3 = \lambda^3 z_1, \ldots$. In general, $z_n = \lambda^{n-1} z_1$. Since z_1 can take any value, every $\lambda \in \mathbf{F}$ is an eigenvalue with eigenvectors

$$(z, \lambda z, \lambda^2 z, \lambda^3 z, \ldots,)$$

for all $z = z_1 \in \mathbf{F}$. □

(2) (a) Problem 18 on page 95 of Axler.
(b) Problem 19 on page 95 of Axler.

Solution. (a) There are many possible answers: for example, let $T \in \mathcal{L}(F^2)$ have matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with respect to the standard basis. Then T is its own inverse, as is easily checked.

(b) Again, many possible answers. For one, let T have matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $T(1,0) = T(0,1) = (1,1)$, so T is not injective, hence not invertible.

Between these two problems, we see that Axler's theorem 5.16 doesn't work in the general case where T doesn't have an upper-triangular matrix. □

(3)(a) Problem 21 on page 95 of Axler.
(b) Problem 22 on page 95 of Axler. (Hint: see page 92 for notation $p_{U,W}$.)

Solution. (a) First we prove that P and $\operatorname{range} P$ have trivial intersection. For suppose $v \in P \cap \operatorname{range} P$. Then $v = Pw$ for some $w \in V$, but V is also in the null space, so $0 = Pv = P^2w = Pw = v$ and v is the zero vector. Hence $P \cap \operatorname{range} P = 0$, and we can consider the direct sum $P \oplus \operatorname{range} P \subset V$. We claim it is all of V: take any $v \in V$. We can write $v = (v - Pv) + Pv$. The second term is in $\operatorname{range} P$ by definition: we claim that in addition $v - Pv \in P$. Just use linearity: $P(v - Pv) = Pv - P^2v = 0$ by our assumption. So $v - Pv \in P$, $Pv \in \operatorname{range} P$, and their sum v is an element of $P \oplus \operatorname{range} P$. Our vector v was arbitrary, so $V = P \oplus \operatorname{range} P$.

(b) Suppose that v is an eigenvector of $p_{U,W}$, with eigenvalue λ. Our vector space V is the direct sum $U \oplus W$, so we can write v uniquely as $u + w$, where $u \in U$, $w \in W$. Then by definition of the projection map, $Tv = u$: but λ is an eigenvector, so $Tv = \lambda v = \lambda u + \lambda w$ as well. Because $V = U \oplus W$, these representations $Tv = u + 0 = \lambda u + \lambda w$ must be identical: that is, $u = \lambda u$, $\lambda w = 0$. Because v is nonzero, u and w aren't both 0.

If u is nonzero, the first equation implies $\lambda = 1$, so $w = 0$, and $v = u \in U$. Any $v = u \in U$ clearly satisfies $p_{U,W}v = 1 \cdot v$, so the eigenvectors of $p_{U,W}$ with eigenvalue 0 are exactly the elements of U.

If w is nonzero, the second equation implies $\lambda = 0$, so $u = 0$, and $v = w \in W$. Any $v = w \in W$ clearly satisfies $p_{U,W}v = 0 \cdot v$, so the eigenvectors of $p_{U,W}$ with eigenvalue 0 are exactly the elements of W. □

(4) Problem 5 page 122 Axler.

Solution. We'll show that our norm $\|\|$ doesn't satisfy the parallelogram equality. Because this equality holds for any norm coming from an inner product, this will imply that our norm cannot come from an inner product.

The parallelogram law is very badly broken for this norm, so it is easy to find counterexamples. For example, take $u = (1, 0)$, $v = (0, 1)$. Then $\|u\| = 1 + 0 = 1$, likewise, $\|v\| = 1$, and $2(\|u\|^2 + \|v\|^2) = 4$. However $\|u + v\| = \|(1, 1\| = 1 + 1 = 2$, and also $\|u - v\| = \|1, -1\| = 2$. So $\|u + v\|^2 + \|u - v\|^2 = 8$. Because $8 \neq 4$, the parallelogram law fails, and we know we are not dealing with a norm that came from an inner product. □

(5) Problem 10 page 123 of Axler. (Please make a note of your answer, you'll need this for HW 12 next week!)

Solution. We apply Gram-Schmidt and crunch the numbers. Let $v_1 = 1$, $v_2 = x$, $v_3 = x^2$. Our first basis vector is

$$e_1 = \frac{v_1}{\|v_1\|} = 1.$$

Our second is

$$e_2 = \frac{v_2 - \langle v_2, e_1 \rangle e_1}{\|v_2 - \langle v_2, e_1 \rangle\|} = \frac{x - \int_0^1 x dx}{\|x - \int_0^1 x dx\|} = \frac{x - \frac{1}{2}}{\sqrt{\int_0^1 (x - \frac{1}{2})^2 dx}} = \sqrt{3}(-1 + 2x)$$

Our third is

$$
\begin{aligned}
e_3 &= \frac{v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2}{\|v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2\|} \\
&= \frac{x^2 - \int_0^1 x^2 dx - \sqrt{3}{-1} + 2x \int_0^1 \sqrt{3}(-1+2x)x^2 dx}{|x^2 - \int_0^1 x^2 dx - x\int_0^1 x^3 dx|} \\
&= \frac{x^2 - x + \frac{1}{6}}{\sqrt{\int_0^1 (x^2 - x + \frac{1}{6})^2 dx}} = \sqrt{5}(1 - 6x + 6x^2).
\end{aligned}
$$

□

(6) Problem 17 on page 124 of Axler.

Solution. Suppose we have such a $P \in \mathcal{L}(V)$ such that every vector of P is orthogonal to every vector of range P and $P^2 = P$. We claim that this P must be the same as the orthogonal projection P_W onto the subspace $W = \operatorname{range} P$, that is, that $Pv = P_W v$ for all $v \in V$. We can apply problem 3 here, to write any $v \in V$ as $v = u + w$, $u \in P$, $w \in \operatorname{range} P$. Then $Pv = Pu + Pw = Pw$. Furthermore, $w \in \operatorname{range} P$, so we can write $w = Pw'$, for some $w' \in V$. So $Pv = Pw = P^2 w' = Pw' = w$.

On the other hand, u, as an element of P, is by assumption orthogonal to anything in range P, so $u \in \operatorname{range} P^\perp$, $w \in \operatorname{range} P$. So our expression $v = u + w$ writes v as the sum of a vector in $W^\perp$ and a vector in W. By definition of orthogonal projection, $P_U v$ is equal to the component belonging to W. That is, $P_U v = w = Pv$ for any $v \in V$, so $P = P_U$ is an orthogonal projection. □