Math 25a Homework 11

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1 Ivan's problems

Note: I've used Nike Sun's pset as the basis for these solutions since she did a very nice job on it!

(1) *Products and coproducts of vector spaces*

Let V_i be a vector space over a field F. For each $i \in I$ where I is some indexing set (possibly infinite) we define

$$\prod_{i\in I} V_i = \left\{ f : I \to \bigcup_{i\in I} V_i \,|\, f(i) \in V_i \ \forall i \right\},$$

$$\coprod_{i\in I} V_i = \left\{ f : I \to \bigcup_{i\in I} V_i \,|\ f(i) \in V_i \, \text{and} \, f(i) = 0 \in V_i \text{ for all but finitely many } i \right\}.$$

We put a vector space structure on $\prod_{i\in I} V_i$ as follows. If $f_1, f_2 \in \prod_{i\in I} V_i$ and $\alpha \in F$, then define $f_1 + f_2 \in \prod_{i\in I} V_i$ by $(f_1 + f_2)(i) = f_1(i) + f_2(i)$ and define $\alpha f_1 \in \prod_{i\in I} V_i$ by $(\alpha f_1)(i) = \alpha f_1(i)$.

(a) Check that $\prod_{i\in I} V_i$ is a vector space. In fact, just check any two of the vector space axioms and submit for grading. (Check the rest, but don't submit for grading.)

(b) Let $I = \{1, 2, \ldots, n\}$ and $V_i = F$ for all i. Show that $\prod_{i\in I} V_i \cong F^n$. (Do this by constructing a map from $\prod_{i\in I} V_i$ to F^n, checking it is linear and that it is invertible.)

(c) Check that $\coprod_{i\in I} V_i$ is a subspace of $\prod_{i\in I} V_i$.

(d) Show that when I is finite that $\prod_{i\in I} V_i = \coprod_{i\in I} V_i$ and that when I is infinite that $\coprod_{i\in I} V_i \subsetneqq \prod_{i\in I} V_i$. (For the latter part you should also assume that $V_i \neq \{0\}$ for all $i \in I$.)

Solution. (a)We will show commutativity and Associativity. Observe that for f_1 and f_2,

$$(f_1 + f_2)(i) = f_1(i) + f_2(i) = f_2(i) + f_1(i) = (f_2 + f_1)(i).$$

Likewise,

$$((f_1+f_2)+f_3)(i) = (f_1+f_2)(i)+f_3(i) = (f_1(i)+f_2(i))+f_3(i) = f_1(i)+(f_2(i)+f_3(i)) = f_1(i)+(f_2+f_3)(i) = (f_1$$

(b) Define $\phi : \prod_{i\in I} V_i \to F^n$ by $\phi(f) = (f(1), \ldots, f(n))$. Then for $f_1, f_2 \in \prod_{i\in I} V_i$,

$$\phi(f_1+f_2) = ((f_1+f_2)(i), \ldots, (f_1+f_2)(n)) = (f_1(i)+f_2(1), \ldots, f_1(n)+f_2(n)) = (f_1(1), \ldots, f_1(n))+(f_2(n), \ldots$$

Similarly

$$\phi(af) = ((af_1)(1), \dots, (af1)(n)) = (af_1(1), \dots, af_1(n)) = a(f_1(1), \dots, f_1(n)).$$

Thus we have linearity.

Checking it is a bijection is obvious. Thus we are done.

(c) This is clear from the definition.

(d) The first part is straight forward as we are in the finite case, and for the second part consider $f \in \prod V_i$ where $f(i) = 1_{v_i}$ for all i. Clearly $f \notin \coprod V_i$.

□

(3) *Universal mapping properties*
Let V_i be a vector space over a field F and $\prod_{i\in I} V_i$ defined as in (1).

(a) Define a family of linear transformations as follows: let $proj_i : \prod_{i\in I} V_i \to V_i$ be defined by $proj_i(f) = f(i)$ for all $i \in I$.
Show that given *any* vector space W and *any* family of linear transformations $\gamma_i : W \to V_i$ (for all $i \in I$) there exists a unique linear transformation $\gamma : W \to \prod_{i\in I} V_i$ such that

$$proj_i \circ \gamma = \gamma_i \quad \forall i \in I. \qquad\qquad (\#)$$

(Hint: we need to define γ. So for $w \in W$ observe that $\gamma(w) \in \prod_{i\in I} V_i$, that is $\gamma(w)$ is a map. What does it do? Hint: define $\gamma(w)(i) := \gamma_i(w)$ for all $i \in I$. Now check γ is linear, check uniqueness and check property (#).)

Solution. This will be a little messy since I haven't learned diagrams in tex. We have the implied commutative diagram from the question. We define the unique γ from W to $\prod V_i$ and show it is linear and unique and that it meets #.

Define $\gamma : W \to \prod V_i$ such that $w \mapsto \gamma(w)$ where $\gamma(w)(i) := \gamma_i(w)$. This clearly meets #. Uniqueness is easy by projections.and linear it clear for all i. □

The conditions described in part (a) are known as the universal mapping property (UMP) for products. Note: the tuple

$$\left(\prod_{i\in I} V_i \ , \ \{proj_i : \prod_{i\in I} V_i \to V_i\}_{i\in I} \right)$$

is called a *product* of the family $\{V_i\}_{i\in I}$. In part (b) we'll show that this UMP determines products uniquely up to isomorphism.

(b) Suppose that

$$\left(Z \ , \ \{p_i : \prod_{i\in I} V_i \to V_i\}_{i\in I} \right)$$

satisfies the UMP for products. (To repeat, this means that given any vector space W and any family of linear maps $\gamma_i : W \to V_i$ then there exists a unique map $\gamma : W \to Z$ such that $p_i \circ \gamma = \gamma_i$.) Show that $Z \cong \prod_{i\in I} V_i$.

(Hint: Use the UMP for Z and also for $\prod_{i\in I} V_i$ to construct linear transformations $\gamma_1 : Z \to \prod_{i\in I} V_i$ and $\gamma_2 : \prod_{i\in I} V_i \to Z$. Now show $\gamma_1 \circ \gamma_2 = I_{\prod V_i}$ and show $\gamma_2 \circ \gamma_1 = I_Z$. (Note that I_Z is the identity map on Z.) For this last part you will need to use the UMP again, in particular the part about uniqueness.)

Solution. We have $\gamma_1 : Z \to \prod V_i$ and $\gamma_2 : \prod V_i \to Z$, by use of the UMP. We want to show that $\gamma_1 \circ \gamma_2 = I_{\prod V_i}$ and same idea for I_Z. However this must be since we can use the UMP between $\prod V_i$ and itself to get a unique identity which by definition must be the composition. □

(4) *Universal mapping properties continued*
Let V_i be a vector space over a field F and $\coprod_{i\in I} V_i$ defined as in (1).

Define a family of linear transformations as follows: for all $i \in I$ let $inj_i : V_i \to \coprod_{i\in I} V_i$ be defined by $inj_i(v_i) = f_{v_i}$ where f_{v_i} is the map defined by

$$f_{v_i}(j) = \begin{cases} v_i & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

You can show that given any W and a family of linear transformations $\gamma_i : V_i \to W$, there is a unique map $\gamma : \coprod_{i\in I} V_i \to W$ satisfying $\gamma \circ inj_i = \gamma_i$ for all $i \in I$. Namely $\gamma(f) = \sum_{i\in I} \alpha_i(f(i))$. (Note that this sum is well defined as $f(i) = 0$ for all but a finite number of i.)

This is the universal mapping property for coproducts. The tuple

$$\left(\coprod_{i\in I} V_i \ , \ \{inj_i : V_i \to \coprod_{i\in I} V_i\}_{i\in I} \right)$$

is called the coproduct of the vector spaces V_i. Note that the analogue of 2(b) holds — that is, the UMP determines the coproduct up to isomorphism. There is no real exercise for you to do here or to submit. Just think all of this through, make sure you understand what is going on and see if you can imitate the arguments in question 2 in this new setting.

(5) (a) In class we defined $V = V_1 \oplus V_2$ for subspaces V_1 and V_2 of the vector space V. Show that $V_1 \oplus V_2 \cong \coprod_{i\in I} V_i$.

Solution. Consider $\psi : V_1 \oplus V_2 \to \coprod_{i\in I} V_i$ defined by $\psi(v_1 + v_2) \mapsto f \in \coprod V_i$ where $f : I \to \bigcup V_i$ and $f(1) = v_1$ and $f(2) = v_2$. It is clearly linear and bijective. □

(b) Now consider $W = \coprod_{i\in\{1,2\}} V_i$. Define $W_i \subset W$ by

$$W_i := \{f \in \coprod_{j\in I} V_i \mid f(j) = 0 \text{ for } j \neq i\}$$

Show (i) W_i is a subspace of W for all i (ii) $W_i \cong V_i$ and (iii) $W = W_1 \oplus W_2$.

Solution. (i): That W_i is subspace of W is clear.

(ii): $W_i \cong V_i$ as $\phi : W_i \to V_i$ with $f \mapsto f(i)$ is clearly linear and a bijection. □

(iii): Now $W = W_1 \oplus W_2$. Clearly for $f \in W$ we can write it as $f_1 + f_2$ for $f(1) \in V_1$ and $f(2) \in V_2$, and clearly the intersection of W_1 and W_2 is trivial, so we are done.

Solution. □

(A word of caution: For vector spaces we often abuse notation and write $\bigoplus$ and $\coprod$ interchangeably.)