

Math 25a Homework 12 Solutions

Ivan Corwin and Alison Miller.

1 Alison's problems

(1) Problem 12 on page 245 of Axler.

Solution. We know that $\text{tr } T$ equals the sum of the diagonal entries of T : $\text{tr } T = 51 + -40 + 1 = 12$. However, $\text{tr } T$ is also the sum of the eigenvalues of T , so $12 = \text{tr } T = -48 + 24 + \lambda_3$ which implies $\lambda_3 = 36$. \square

(2) Problem 9 on page 244 of Axler.

Solution. From Homework 11 3a) we know that $V = \text{null } P \oplus \text{range } P$. Let $\{v_1, \dots, v_m\}$ be a basis of $\text{null } P$ and $\{v_{m+1}, \dots, v_n\}$ a basis of $\text{range } P$. Together they form a basis of V , so we can look at the matrix of P with respect to this basis. For $i = 1, \dots, m$, $Pv_i = 0$ by definition of the null space. Also note that for any $v \in \text{range } P$, $v = Pw$ for some w so $Pv = P^2w = Pw = v$. In particular, $Pv_i = v_i$ for $i = m + 1, \dots, n$. Hence the matrix of P is all 0's except for 1's in the last $n - m$ diagonal entries. So its trace, the sum of its diagonal entries, equals $n - m$, which is a non-negative integer. \square

(3) (a) Problem 14 on page 245 of Axler.

(b) Problem 15 on page 245 of Axler.

Solution. (a) Counterexample: suppose S is given by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and T is given by the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\text{tr } S = \text{tr } T = 1$, but $ST = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\text{tr}(ST) = 0 \neq 1 = (\text{tr } S)(\text{tr } T)$.

(b) We will prove the contrapositive: if T is nonzero, $\text{tr } ST \neq 0$ for some $S \in \mathcal{L}(V)$. Let $M(T) = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix}$: T is nonzero, so some $a_{i,j}$ is nonzero. Let S be the matrix which has a 1 in the j th row and i th column and a 0 everywhere else. Then

$$ST = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ a_{1,j} & \dots & a_{n,j} \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

where the nonzero entries are all in the i th row. Hence the only nonzero entry on the diagonal equals $a_{i,i}$, and $\text{tr}(ST) = a_{i,i} \neq 0$. □

(4) (a) Problem 16 on page 245 of Axler.

(b) Problem 19 on page 245 of Axler.

Solution. (a) Suppose the matrix of T^*T with respect to the basis e_1, \dots, e_n has entries $\{c_{i,j}\}$. Then for each $i = 1, 2, \dots, n$, $T^*Te_i = \sum_{j=1}^n c_{j,i}e_j$. By orthonormality, $\langle T^*Te_i, e_i \rangle = \sum_{j=1}^n c_{j,i} \langle e_j, e_i \rangle = c_{i,i}$. Also $\langle T^*Te_i, e_i \rangle = \langle Te_i, Te_i \rangle = \|Te_i\|^2$ by definition of T^* .

Putting it all together, $\text{tr}(T^*T) = \sum_{i=1}^n c_{i,i} = \sum_{i=1}^n \langle T^*Te_i, e_i \rangle = \sum_{i=1}^n \langle \|Te_i\|^2 \rangle$.

(b) We will use part (a) to show that in fact $\|Tv\| = \|T^*v\|$ for all $v \in V$. For any $v \in V$, normalize v to the unit vector $e_1 = \frac{v}{\|v\|}$. By Gram-Schmidt, we can extend e_1 to an orthonormal basis $\{e_1, e_2, \dots\}$ of V . By (a),

$$\sum_{i=1}^n \langle \|Te_i\|^2 \rangle = \text{tr}(T^*T) = \text{tr}(TT^*) = \sum_{i=1}^n \langle \|T^*e_i\|^2 \rangle.$$

By our hypothesis, we know that each left hand side term is less than or equal to the corresponding right hand term: so corresponding terms must all be equal for equality to hold. In particular, $\|Te_1\|^2 = \|T^*e_1\|^2$. Multiplying both sides by $\|v\|^2$, we see that $\|Tv\|^2 = \|T^*v\|^2$. This is true for all v , so T is normal by Proposition 7.6. □

(5) (a) Problem 20 on page 245 of Axler.

(b) Problem 21 on page 245 of Axler.

Solution. (a) Proof: $\det cT = \det((cI)T) = (\det cI)(\det T)$. The operator cI has all eigenvalues equal to c , so $\det(cI) = c^{\dim V}$. The desired formula follows.

(b) Counterexample: Let $S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then both S and T have rows that are all 0, so $\det S, \det T = 0$. On the other hand, $S + T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which has determinant $\det(S + T) = 1 \neq 0 = \det S + \det T$. □