Math 25a Homework 13

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1 Ivan's problems

(1) (a) Problem 6 on page 122 of Axler (so back to Ch 6).

Solution. Observe that

$$\frac{1}{4}(||u+v||^2 - ||u-v||^2) = \frac{1}{4}(\langle u+v, u+v\rangle - \langle u-v, u-v\rangle$$

and by expanding and canceling terms we get that equals

$$\frac{1}{4}(2\langle v, u\rangle + 2\langle u, v\rangle).$$

Since we are in a real inner product space, the two terms are both real and hence $\langle v, u\rangle = \overline{\langle u, v\rangle} = \langle u, v\rangle$ giving the desired result. $\square$

(b) Take a look at Problem 7 on page 122 of Axler. Don't hand this one in. The computations are similar, but longer than Problem 6. I meant to set these problems earlier. These reformulations of the inner product can be quite handy.

(2) Recall from HW 12 that we defined the tensor product of two vector spaces. Stop right now, get out your old assignment and take a look at all the notation we covered. I remarked at the end of Problem 5, that the tensor product of V and W is usually denoted $V \otimes W$ and the image $\mu((v, w)) = [\delta_{(v,w)}]$ is generally written as $v \otimes w$. Make sure that you understand that in this notation $(a_1v_1 + a_2v_2) \otimes w = a_1(v_1 \otimes w) + a_2(v_2 \otimes w)$ and $v \otimes (b_1w_1 + b_2w_2) = b_1(v \otimes w_1) + b_2(v \otimes w_2)$. (In case you wondered about the peculiar definition of Z in the previous HW, note that it is this very definition that gives tensors these nice properties.)

Now let V and W be vectors spaces over a field F and let V have basis $R = \{v_1, \ldots, v_n\}$ and let W have basis $S = \{w_1, \ldots w_m\}$. The aim of this question is to show that $B = \{v_i \otimes w_j | 1 \leq i \leq n, 1 \leq j \leq m\}$ is a basis for $V \otimes W$.

(a) Show that B spans $V \otimes W$.

Solution. For any $v \in V$ and $w \in W$, we may express each one, respectively, in terms of the basis for V and W. Let $v = \sum a_i v_i$ and $w = \sum b_j w_j$. Then $v \otimes w = (\sum a_i v_i) \otimes (\sum b_i w_i) = \sum \sum a_i b_j (v_i \otimes w_j)$. So every element in $V \otimes W$ is expressible in terms of the basis $v_i \otimes w_j$. □

(b)(i) Let

$$U = \coprod_{R \times S} F.$$

Show that $\{f_{ij} : R \times S \to F\}$ is a basis for U where f_{ij} is defined as follows:

$$f_{ij}((v_k, w_l)) = \begin{cases} 1 & \text{if } k = i \text{ and } l = j \\ 0 & \text{otherwise.} \end{cases}$$

Solution. Since R and S are finite $U = \{f : R \times S \to F | f((v_k, w_l)) \in F$. For every $f \in U$, f acts by taking a pair (v_i, w_j) to the scalar $a_{ij} \in F$. So f can be decomposed as $f = \sum \sum a_{ij} f_{ij}$. Therefore the $\{f_{ij}\}$ span U. Linear independence follows since removing and f_{ij} results in (v_i, w_j) going to 0. Therefore we have a basis for U. □

(b)(ii) Show that the map $\phi : V \times W \to U$ defined by $\phi(\sum_i a_i v_i, \sum_j b_j w_j) := \sum_{i,j} a_i b_j f_{ij}$ is bilinear.

Solution. Consider $\phi(\alpha_1 a + \alpha_2 b, w)$ where $a, b \in V$. Clearly we can express a and b in terms of the basis for V and collect the basis terms until we have an equation of the form $\phi(\sum blah, \sum blah_2)$. Then we can use the definition of ϕ and undo all of our computations to get the linearity. Symmetry applies to get bilinearity. □

(b)(iii) Show that $B = \{v_i \otimes w_j | 1 \leq i \leq n, 1 \leq j \leq m\}$ is a linearly independent set in $V \otimes W$.

Solution. From UMP we have a unique linear map $\hat{\phi} : V \otimes W \to U$ with $\hat{\phi} \circ \mu = \phi$ where $\mu((v, w)) = v \otimes w$. Now consider a linear dependence $\sum a_{ij} w_i \otimes w_j = 0$ and apply $\hat{\phi}$ to both sides giving eventually $0 = \sum a_{ij} f_{ij}$, but this is a contradiction since we have shown the f_{ij} are linearly independent. Thus B is a linearly independent set in $V \otimes W$. □

(3) (a) If V is a finite dimensional space, and if u and v are in V, it is true that $u \otimes v = v \otimes u$?

Solution. Consider $u, v \in W$. If $u \neq v$ then $u \otimes v = [\delta_{(u,v)}]$ and $v \otimes u = [\delta_{(v,u)}]$ since their difference is not in Z (the thing we mod out by). Thus the statement does not hold. □

(b) Now let x, y be a basis for the vector space V over the field F. Using Problem 2 we know that every element of $V \otimes V$ can be written uniquely in the form

$$ax \otimes x + bx \otimes y + cy \otimes x + dy \otimes y$$

for some $a, b, c, d \in F$. Firstly, show that $x \otimes y + y \otimes x$ is not equal to $v \otimes w$ for any choice of $v, w \in V$. Secondly, for what values of a, b, c, d can the tensor $ax \otimes x + bx \otimes y + cy \otimes x + dy \otimes y$ be written as $v \otimes w$?

Solution. We can write every $v \in V$ as $v = \alpha x + \beta y$ so

$$v_1 \otimes v_2 = (\alpha_1 x + \beta_1 y) \otimes (\alpha_2 x + \beta_2 y) = \alpha_1 x \otimes (\alpha_2 x + \beta_2 y) + \beta_1 y \otimes (\alpha_2 x + \beta_2 y) = \alpha_1\alpha_2 x \otimes x + \alpha_1\beta_2 x \otimes y + \beta_1\alpha_2 y \otimes x + \beta$$

This can not equal $x \otimes y + y \otimes x$ since then $\alpha_1\alpha_2 = \beta_1\beta_2 = 0$ but also $\alpha_1\beta_2 = \beta_1\alpha_2 = 1$, a contradiction. In general, letting $a = \alpha_1\alpha_2$, $b = \alpha_1\beta_2$, $c = \beta_1\alpha_2$ and $d = \beta_1\beta_2$ we must have $ad - bc = 0$. □

(4) Suppose that V and W are finite dimensional vector spaces over a field F. Recall that we proved that the set

$$\mathcal{L}(V, W) = \{T : V \to W : T \text{ is linear}\}$$

forms a vector space. (See Axler page 40.) Construct an isomorphism between $\mathcal{L}(V, W)$ and $V^* \otimes W$ where V^* is the dual space to V. (In fact, this isomorphism is canonical — it will not depend on the bases chosen.) (Hint: as useful as it is, don't just count dimensions. I want to see an actual map. There are at least two ways to do this problem. One of these ways uses the knowledge you've gained from Problem 2.)

Solution. Consider a basis for V, $\{v_i\}$ and the dual basis for V^* of $f^i : V \to F$ such that $f^i(v_j) = \delta_{ij}$. By problem 2, $V^* \otimes W$ has a basis of $f^i \otimes w_j$. Now let $\phi' : B \to Hom(V, W)$ be defined by $\phi'(f^i \otimes w_j)(v) = f^i(v) w_j$. By linearity this extends to a map ϕ on all of $V^* \otimes W$.

Now let us show this is an isomorphism. Observe that if $\phi(f \otimes w)(v) = 0$ then expanding we find that $\phi(f \otimes w)(v) = \phi(\sum a_{ij} f^i \otimes w_j)(v) = \sum a_{ij} f^i(v) w_j$ and hence all of the $a_{ij} = 0$, so the preimage was zero. This gives us injectivity. For surjectivity observe that any element in the range can be thought of as a matrix given our basis. Then any such matrix $\{a_{ij}\}$ can be mapped to by the element in the domain, $\sum a_{ij} f^i \otimes w_j$. Thus we have an isomorphism. □