

Math 25a Homework 4 Solutions

Ivan Corwin and Alison Miller.

1 Alison's problems

(1) Problem 17 on page 44 of Rudin.

Solution. E is not countable: If E were countable, we could enumerate all its elements as e_1, e_2, \dots . Then, letting $a^{(k)}$ denote the k th decimal digit (after the decimal point) of a : Let d be such that $d^{(k)} = 11 - e_k^{(k)}$. Then, $d \neq e_k$ for any k , for $d^{(k)} \neq e_k^{(k)}$. So, our enumeration is incomplete. This yields a contradiction, and so we see that E is not countable.

E is not dense in $[0, 1]$: Take $x = 0.5$ and $r = 0.01$, then $N_r(x) = (0.49, 0.51)$ clearly contains no elements of E (as $0.4\bar{7}$ is the greatest element in E less than $.5$, and $0.7\bar{4}$ is the greater than $.5$).

E is compact: E is clearly bounded (as a subset of $[0, 1]$). Furthermore, E is closed, for

$\mathbf{R} \setminus E$ is open (for any $x \notin E$, it will have some first digit in its decimal representation that is not a 4 or a 7, and then we can take a neighborhood of x such that it is entirely in $\mathbf{R} \setminus E$ by ensuring that this digit doesn't become a 4 or a 7). Then, by the Heine-Borel theorem, E is compact.

E is perfect: For any element $e = .e^{(1)}e^{(2)}e^{(3)} \dots$, we can take the sequence e_1, e_2, \dots such that e_k differs from e only in the k th digit (which goes from x to $11 - x$). Then, (e_k) converges to e , as $|e - e_k| \leq 10^{1-k}$. \square

(2) Problem 19 on page 44 of Rudin.

The definition of separated sets in on page 42 of Rudin.

Solution.

- (a) We note that $\bar{A} = A, \bar{B} = B$, so as A, B are disjoint we have that $A \cap \bar{B} = \bar{A} \cap B = A \cap B = \emptyset$.

- (b) WLOG say $\bar{A} \cap B \neq \emptyset$, then there is some $x \in \bar{A} \cap B$. So, there exists some $r > 0$ such that $N_r(x) \subset B$ (for B is open), but $N_r(x) \cap A \neq \emptyset$ as x is a point or limit point of A . But, A and B are disjoint, so this yields a contradiction, and A and B must be separated.
- (c) Note that A is open, as for any $x_0 \in A$ we have $d(p, x_0) < \delta$, so for $r_0 < \delta - d(p, x_0)$ we have that (by the triangle inequality) $N_{r_0}(x_0) \subset A$. Similarly, B is open, as for any $x_0 \in B$ we have $d(p, x_0) > \delta$, so for $r_0 < d(p, x_0) - \delta$ we have that $N_{r_0}(x_0) \subset B$. Then, we can apply the previous part.
- (d) Say X were a connected at most countable metric space of at least 2 points. For each point $x \in X$, let $D_x = \{d(p, x) | p \in X, p \neq x\}$. There must exist some $x_0 \in X$ such that $|D_{x_0}| > 2$ [for otherwise X carries the discrete topology, and then for any $x_0 \in X$, $\{x_0\}$ and $X \setminus \{x_0\}$ would each be open sets, and thus separated]. Take some such x_0 , and let $m = \inf D_{x_0}$, $M = \sup D_{x_0}$. Then, there must be some $\delta \in [m, M]$ such that $\{p | d(p, x_0) = \delta\}$ is empty – for otherwise, we would have an injective map from $[m, M]$ to X , and so X could not be at most countable. Take this δ , and note that $\{p | d(p, x_0) < \delta\}$ and $\{p | d(p, x_0) > \delta\}$ are disjoint non-empty sets partitioning X , and by (c) are separated, contradicting the connectedness of X .

□

(3) Problem 20 on page 44 of Rudin.

Solution. For closures: *Yes.* Let A be a connected set, and say $\bar{A} = C \cup D$ with C, D separated and disjoint. Then, consider $C' = A \cap C$, $D' = A \cap D$. Clearly, C', D' are disjoint and separated, for $\bar{C}' \subset \bar{C}$ and similarly for D . Also, say one of C', D' is empty. WLOG, say C' is empty. Then, $C \subseteq \bar{A} \setminus A \subseteq A'$, but then $A \subseteq D$, so $A' \subseteq \bar{A} \subseteq \bar{D}' \subseteq D$, so C and \bar{D} will have non-empty intersection, violating the fact that C and D were separated. This yields a contradiction, and so A is connected.

Note: A lot of you made the same mistake here: be careful! To show that \bar{A} is connected, you need to be able to show that however you write $\bar{A} = C \cup D$, C and D are not separated. People tried to say that if $A = C \cup D$, C and D are not separated by definition, so $\bar{C} \cup \bar{D}$ is not a separation of \bar{A} either – however, this doesn't show that \bar{A} can't have some other separation. Keep in mind: pairs of sets are separated; single sets are connected. (Also,

you might like to think about why it is important to the proof that \bar{A} is the closure of A and not some random set containing A .) For interiors: *No*. Let $A = \{(x, y) | (x + 1)^2 + y^2 \leq 1\}$, $B = \{(x, y) | (x - 1)^2 + y^2 \leq 1\}$: we take our connected set to be the union $S = A \cup B$. We first notice that the two disks A and B intersect only at the origin, where they are tangent. Suppose for a contradiction that we had a separation of S into sets C, D . WLOG, assume that the origin $(0, 0) \in C$. Then D is nonempty, so let d be an arbitrary element of D : this point must lie in one of the two disks A, B , both of which are convex. This means that the line segment L with endpoints 0 to d lies in D . We know that line segments (i.e. closed intervals) are connected in \mathbb{R} , so they remain connected if we look at them as subsets of the larger space \mathbb{R}^2 . (As for compactness, it doesn't matter which set you are working, as long as you keep the same metric.) However, this raises a problem, because we can write $L = (L \cap C) \cup (L \cap D)$, which is a separation, because $L \cap C$ and $L \cap D$ are subset of the separated sets C and D (and both nonempty, because the first one contains 0 and the second contains D). So we've got a contradiction, and can conclude that S is connected.

However, $\overset{\circ}{S}$ is not connected. You can show without too much trouble (but you need to take care to the detail!) that this interior is just the union of the interiors of the two disks, but we won't actually use that here. We'll just show that this interior does not contain the origin: this is clear, because for any $r > 0$, the neighborhood $N_r(0)$ contains the point $(0, r/2)$, which doesn't lie in either disk. Also, neither disk contains any other point on the y -axis, so $\overset{\circ}{S}$ doesn't contain any points on the y -axis. We'll use this to split $\overset{\circ}{S}$ in half down the y -axis. We define open half planes $H_1 = \{(x, y) \in \mathbb{R}^2 | x > 0\}$, $H_2 = \{(x, y) \in \mathbb{R}^2 | x < 0\}$. We should check that these are actually open: for any $x \in H_1$, the neighborhood $N_x(x, y)$ is contained in H_1 , and the same is true in H_2 , so they are both open. Let $C = \overset{\circ}{S} \cap H_1$, $D = \overset{\circ}{S} \cap H_2$: $\overset{\circ}{S}$ is the disjoint union of $C \cup D$ because it has no points on the y -axis. Also, for any $x \in C$, x is contained in the open half-plane H_1 . Hence x has a neighborhood contained in H_1 , which must be disjoint from D . So x cannot be either a point or a limit point of D , and $C \cap \bar{D}$ is empty. Similarly, $D \cap \bar{C}$ is empty, and the two sets are separated. (You can also proving this by using that the sets C and D are the interiors of the sets A and B , so they are disjoint open sets, which must be separated by the previous problem.)

(A couple of you also found the example of $S =$ union of the closed first quadrant and closed third quadrant in \mathbb{R}^2 . This is in some sense slightly nicer to work with.) \square

(4) Problem 23 on page 45 of Rudin.

The definition of separable is in problem 22 on page 45 of Rudin.

Let S be a countable dense subset of X . Let $\mathcal{B} = \{N_q(s) \mid q \in \mathbb{Q}^{>0}, s \in S\}$. Because both S and \mathbb{Q} are countable, this is a countable union of countable collections of neighborhoods, hence it is itself countable. We now need to check that \mathcal{B} is a base. Let x be an arbitrary point of X and G an arbitrary open set containing x . Because G is open, there is some $r \in \mathbb{R}$ such that $N_r(x) \subset G$. By the Archimedean property of the reals, there exists a positive integer n such that $1/n < r/2$. Because S is dense, there is some $s \in S$ such that $s \in N_{1/n}(x)$, so $d(s, x) < 1/n$. Consider the neighborhood $N_{1/n}(s)$, which is an element of our collection \mathcal{B} . Clearly, $x \in N_{1/n}(s)$. Furthermore, if $y \in N_{1/n}(s)$, by the triangle inequality $d(y, x) \leq d(y, s) + d(s, x) < 1/n + 1/n < r$, so $y \in N_r(x)$. This means that $x \in N_{1/n}(s) \subset N_r(x) \subset G$, which is exactly what we needed to show. So \mathcal{B} is in fact a countable base.

Note: This is all very general and abstract. For a specific example, think about $X = \mathbb{R}$, $S = \mathbb{Q}$. Then \mathcal{B} becomes the set of all balls with rational center and rational radius, or what comes to the same thing, the set of all intervals with rational endpoints.

(5) Problem 25 on page 45 of Rudin.

The hint makes this problem straightforward to do, but it is an important result!

Solution. Let $O_n = \{N_{\frac{1}{n}}(x) \mid x \in K\}$. Then, let S_n be a finite subcover of O_n for each n . Then, let $V = \bigcup_{n \in \mathbb{N}} S_n$, which is clearly countable. To see that V is a base for K , take any $x \in G$, G open, then there exists an $r > 0$ such that $N_r(x) \subset G$; take N such that $\frac{1}{N} < \frac{r}{2}$, then let $V' \in V$ be the element of S_N that contains x , and note that $V' \subset N_r(x) \subset G$, by the triangle inequality, so V is a countable base for K .

List the elements of V in a sequence $\{V_n\}$. For each n , pick an element a_n of V_n . (This uses the Axiom of Choice, but that's okay: we like the Axiom of Choice.) Then the set of the $\{a_n\}$ is countable. Now we just need dense. That is, for any $x \in K$, we need to show that any neighborhood $N_r(x)$ of x contains some a_n . But V is a base, so there exists a $V_n \in V$ such that $x \in V_n \subset N_r(x)$. But $a_n \in V_n \subset N_r(x)$, so we're happy, and K is dense. □

2 Ivan's problems

(1) Problem 26 on page 45 of Rudin.

You may assume the results of problems 23 and 24 on page 45 of Rudin.

Solution. Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable open base (which exists by Rudin problems). Then, let $\{V_\alpha\}_{\alpha \in A}$ be any open cover of X . We know that each V_α is the union of some U_j 's (just take some U_i for each point in V_α , so that $U_i \subset V_\alpha$, and union them all). For each U_j that is in one of these unions, take V_{a_j} where V_{a_j} was one of the V_α whose union U_j was in. Let $G_n = V_{a_j}$. Then $\{G_n\}_{n \in \mathbb{N}}$ is a countable subcover of X (countable as it has only elements of $\{U_i\}_{i \in \mathbb{N}}$, open as all the sets are open, and a subcover as we've included all the U_j 's that union to our original cover).

Now, consider $S_n = \bigcup_{i=1}^n G_i$. If some $S_i = X$, then we have a finite subcover. Otherwise, letting $\bar{S}_i = F_i$, $F_i = X - S_i \neq \emptyset$. Remember however, that $\bigcup_{i \in \mathbb{N}} G_i = X$, and so $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$. Then, by the axiom of choice, let E be a set which contains an element from each F_i (note that we have an infinite number of points, or else there must be some fixed value in each F_n , $n > N$ for some N). Now, E is an infinite set, and so has a limit point, $x \in X$. Also, since $F_{n+1} \subset F_n$, and by the definition of E , for each F_i , at most finitely many points of E are not in F_i . Then, x is also a limit point of F_i for each i . But, $F_i = \bar{S}_i$, and as S_i was open, is closed. Thus, $x \in F_i$ for each i . But then, $\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$. This yields a contradiction. Thus, we must have a finite subcover for any cover, and X is compact. \square

(2) Problem 27 page 45 of Rudin.

Solution. We know R^k has a countable base $\{V_n\}$. Let W be the union of V_n for which $E \cap V_n$ is at most countable. Take $p \in E$. If $p \in W$, then $p \in V_n$ for some n . Since V_n is open and $V_n \cap E$ is at most countable, p is contained in some neighborhood with at most countable points of E . Then p is not a condensation point of E , a contradiction. So $P \subset W^c$.

If p is not a condensation point of E , then there exists an open neighborhood of p with at most countable points of E . But this neighborhood contains an element V_n of the base of R^k . So $V_n \cap E$ is at most countable, and $p \in W$. So $P^c \subset W \Rightarrow W^c \subset P$.

By the above, $P = W^c$, and $P^c \cap E = W \cap E$ is an at most countable union of at most countable sets, and so is at most countable.

To see P is perfect, we need to show that P is closed and every point is a limit point of P . Since $P^c = W$, which is a union of open sets and is

therefore open, P is closed. Consider $p \in P$. Take a neighborhood $N_r(p)$ around p . It has uncountably many points of E . Since $P^c \cap E$ is at most countable, $P^c \cap N_r(p)$ has at most countable many elements of E . So there is a point in $N_r(p)$ not in P^c , which means it is in P . Therefore, p is a limit point. \square

Now read through problem 28. This result is an immediate corollary of problem 27. You do not need to hand in problem 28 - just read and appreciate it.

(3) Let K be a compact subset of the metric space (X, d_X) and let L be a compact subset of the metric space (Y, d_Y) . The *product metric* on the set $X \times Y$ is

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

This is a metric on the set $X \times Y$ (you don't need to prove this). Show that $K \times L$ is a compact subset of $X \times Y$.

Solution. Take a sequence $\{(x_i, y_i)\} \subset K \times L$. The set $\{x_i\}$ has a converging subsequence $\{x'_i\}$ to a limit point x . Take the subsequence of $\{(x_i, y_i)\}$ where the first component is in $\{x'_i\}$. The corresponding second components $\{y'_i\}$ has some limit point y . Now take the subsequence of $\{(x'_i, y'_i)\}$ where the second component converges to y . Now for any ϵ , we can easily find points beyond which $d_x(x_i, x) < \epsilon/2$ and $d_y(y_i, y) < \epsilon/2$, and thus $d((x_i, y_i), (x, y)) < \epsilon$ after some easy algebra. So every sequence in $K \times L$ has a limit point in it, and it is compact. \square

Optional: What does this problem tell you about k -cells (section 2.17 of Rudin) and \mathbb{R}^k ? To answer this, you'll have find a different looking product metric.

(4) Problem 30 on page 46 of Rudin.

Solution. We first formulate the equivalent statement: If G_n is a dense open subset of \mathbb{R}^k , then $\bigcap \{G_n\}$ is not empty. The equivalence can be seen by taking G_n to be the complements of F_n and taking the contrapositive of the original problem statement.

First, we show that $G_1 \cap G_2$ is open and non-empty. Consider $x \in G_1$. Either $x \in G_2$ or x is a limit point of G_2 . We are done in the former case. In the latter, $\exists N_r(x) \in G_1$ and $q \in N_r(x)$, $q \in G_2$. So $q \in G_1 \cap G_2$.

$G_1 \cap G_2$ is open, so take a neighborhood $N_r(y) \in G_1 \cap G_2$. Take $V_1 = \overline{N_{r/2}(y)}$. V_1 is compact. We now construct V_i inductively. Since G_3 is dense, either $y \in G_3$ or y is a limit point of G_3 . In the first case, we may take a neighborhood $N_{r'}(y)$ entirely within $V_1 \cap G_3$. In the second case, we find another point $z \in G_3$ at most $r/4$ away from y and construct a neighborhood around z instead.

Repeating this process, we get a nested sequence of V_i , so their intersection contains a point. But $V_i \subset G_1 \cap G_2 \dots \cap G_i$, so the intersection of G_i is also non-empty. \square