Math 25a Homework 5 Solutions

Ivan Corwin and Alison Miller.

1 Alison's problems

(1) Find an example of a sequence $\{p_n\}$ in $\mathbb{R}$ such that (i) $\{p_n\}$ is bounded (ii) $|p_{n+1} - p_n| \to 0$ as $n \to \infty$, but which is still not convergent.

Solution. We let our sequence p_n be the sequence

$$\{0, 1, 1/2, 0, 1/3, 2/3, 1, 3/4, 2/4, 1/4, 0, 1/5, 2/5, 3/5, 4/5, 1, 5/6, \dots\}$$

where for each positive n we list the fractions with denominator n in increasing order when n is odd, and in decreasing order when n even. (You can work out a closed form for this as an exercise, if you wish.) Then p_n is not Cauchy, because we can let $\epsilon < 1$. Then for any positive integer N, there are integers $n, m > N$ such that $p_n = 0$ and $p_m = 1$, so $|p_n - p_m| = 1 > \epsilon$.

On the other hand, for any n, $|p_{n+1} - p_n| = \frac{1}{k}$ for some k, and as $n \to \infty$, $k \to \infty$ also, so $|p_{n+1} - p_n| \to 0$. $\qquad\square$

Note: There are many different possible sequences one could use to answer this question, and some of them are trickier to justify than others. This one is one of the simplest ones.

(2) Problem 3 on page 78 of Rudin.

Solution. First note that $s_n < 2 \Rightarrow 2 + s_n < 4 \Rightarrow s_{n+1} = \sqrt{2 + \sqrt{s_n}} \leq 2$. Thus, as $s_1 = \sqrt{2} < 2$, we have, by induction that $s_n < 2$ for all $n \in \mathbf{N}$. Thus, the sequence is bounded above.

Now, also note that $s_2 = \sqrt{2 + s_1} > \sqrt{2} = s_1$. Now, note that $s_{n-1} < s_n \Rightarrow \sqrt{s_{n-1}} < \sqrt{s_n} \Rightarrow \sqrt{2 + \sqrt{s_{n-1}}} < \sqrt{2 + \sqrt{s_n}} \Rightarrow s_n < s_{n+1}$. So by induction, the sequence is increasing.

Then, our sequence is monotonic increasing and bounded above, and thus converges. $\qquad\square$

(3) Problem 5 on page 78 of Rudin.

Solution. First we do the case where both $\limsup a_n$, $\limsup b_n$ are real. For each positive integer N, let

$$
\begin{aligned}
A_n &= \sup\{a_n, a_{n+1}, a_{n+2}, \dots\} \\
B_n &= \sup\{b_n, b_{n+1}, b_{n+2}, \dots\} \\
C_n &= \sup\{a_n + b_n, a_{n+1} + b_{n+1}, a_{n+2} + b_{n+2}, \dots\}
\end{aligned}
$$

We claim that $C_n \leq A_n + B_n$ for any n. To this end, we note that for $k \geq n$, $a_k \leq A_n$, $b_k \leq B_n$, so $a_k + b_k \leq A_n + B_n$. This means that $A_n + B_n$ is an upper bound for the set $\{a_n + b_n, a_{n+1} + b_{n+1}, a_{n+2} + b_{n+2}, \dots\}$, so $A_n + B_n \geq C_n$ by definition of C_n as the supremum.

Now, the definition of $\limsup$ is that $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} A_n$, $\limsup_{n\to\infty} b_n = \lim_{n\to\infty} B_n$ and $\limsup_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} C_n$. Using $A_n + B_n \geq C_n$ for all n, and taking limits,

$$
\begin{aligned}
\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n &= \lim_{n\to\infty} A_n + \lim_{n\to\infty} B_n \\
&= \lim_{n\to\infty}(A_n + B_n) \\
&\geq \lim_{n\to\infty} C_n = \limsup_{n\to\infty} c_n.
\end{aligned}
$$

Now we think about the infinite cases. If either of $\limsup a_n$, $\limsup b_n$ are ∞, the inequality is trivial. So suppose that $\limsup a_n = -\infty$: by assumption, $\limsup b_n < \infty$, so b_n is bounded above by some real number β. Hence the series of supremums B_n is also bounded above by the same real number β. This means that for all n, $C_n \leq A_n + B_n \leq A_n + \beta$. Because $\lim_{n\to\infty} A_n = -\infty$, $\lim_{n\to\infty} C_n = -\infty$ as well. We're now done because $-\infty + B_n = -\infty$. $\qquad\square$

Note: The lim sup notation is a little confusing: some people rewrote $\limsup_{n\to\infty} a_n$ as $\lim_{n\to\infty}(\sup a_n)$. This is sort of unclear because you don't know which set of a_n you are taking the supremum of. Also, the n is doing double duty here: the proper way to write $\limsup a_n$ as a limit of supremums would be as $\lim_{n\to\infty}(\sup_{k\geq n} a_k)$, using two different variables to index the limit and supremum. Bottom line: the lim sup of a set is, as the terminology suggests, limit of supremums, but it's not obvious which limit of supremums it is.

(4) Let $\{p_n\}$ be any bounded sequence of real numbers and let $l = \liminf p_n$ and $u = \limsup p_n$. Prove that $l \leq u$ and moreover, that $l = u$ if and only if $\lim_{n\to\infty} p_n$ exists, in which case $\lim_{n\to\infty} p_n = l = u$.

Proof. As done in class, we let $E_n = \{p_n, p_{n+1}, p_{n+2}, \dots\}$. Then $l = \lim_{n \to \infty} \inf E_n$ and $u = \lim_{n \to \infty} \sup E_n$. Now $\inf E_n \leq \sup E_n$ for all n, so, taking the limit gives us $l \leq u$.

Now assume $l = u$. Then for all n, $p_n \in E_n$, so $\inf E_n \leq p_n \leq \sup E_n$. Using the Squeezing lemma, we see that $\lim_{n \to \infty} p_n = l = u$.

In the other direction, suppose that $\{p_n\} \to p$ for some p. Choose an $\epsilon > 0$ and take an N such that $n \geq N$ implies $|p_n - p| < \epsilon/2$. Then for any $n \geq N$, for all $p_k \in E_n$, $p - \epsilon/2 < p_k < p + \epsilon/2$. Hence $p + \epsilon/2$ is an upper bound for E_n: by definition of supremum, $\sup E_n \leq p + \epsilon/2$. Also, $\sup E_n \geq p_n > p - \epsilon/2$, so $\sup E_n \in (p - \epsilon/2, p + \epsilon/2]$. Hence $|E_n - p| < \epsilon$ for all $n \geq N$. This means that $\{\sup E_n\} \to p$, so $p = \lim \sup p_n = l$. The exact same argument shows $p = u$, so $l = u$. $\qquad\square$

(5) Problem 12 on page 79 and 80 of Rudin.

Solution. (a) The left-hand side is clearly $> 1/r_m(a_m + \dots a_n) = 1/r_m(r_m - r_n) = 1 - r_n/r_m$.

If $\Sigma a_n/r_n$ converges, it is Cauchy. So for any $\epsilon > 0$, we can find N s.t. whenever $n \geq m \geq N$, $(a_m/r_m + \dots a_n/r_n) < \epsilon$. Choose n s.t. $r_n < r_N(1 - \epsilon)$ which is possible since r_n approaches 0. Then we have a contradiction with the above inequality.

(b) $a_n = r_n - r_{n+1} = (\sqrt{r_n} + \sqrt{r_{n+1}})(\sqrt{r_n} - \sqrt{r_{n+1}}) < 2\sqrt{r_n}(\sqrt{r_n} - \sqrt{r_{n+1}})$.
Now, this sum telescopes and can easily be seen to be convergent. $\qquad\square$

2   Ivan's problems

(2) This question is set in $\mathbb{R}$. Let $\alpha > 0$ and let x_0 arbitrary and define a sequence x_n recursively by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

Show that $x_n \to \sqrt{\alpha}$. (Note this is Newton's method for solving $x^2 - \alpha = 0$.)

Solution. We assume all x_n positive. Clearly for $x_0 = \sqrt{\alpha}$ all $x_n = \sqrt{\alpha}$ and we are done. For $x_0 < \sqrt{\alpha}$ simple computation shows $x_1 > \sqrt{\alpha}$. Thus we can assume $x_0 > \sqrt{\alpha}$. Now inductively assume $x_n > \sqrt{\alpha}$. Then $x_{n+1} =$

$(x_n + \alpha/x_n)/2 > \sqrt{x_n * \alpha/x_n} = \sqrt{\alpha}$, with the middle step via AM-GM. So all $x_n \geq \alpha$. Then, since $x_n \geq \sqrt{\alpha}$, $x_{n+1} = (x_n + \alpha/x_n)/2 \leq x_n$. Thus x_n is monotonically decreasing and bounded below. Therefore x_n converges to some x.

It remains to show that $x = \sqrt{\alpha}$. This can be done either through an epsilon argument or through limits. The epsilon argument says, assume it converges to some $x > \sqrt{\alpha}$, then for large n, x_n is within an ϵ of x which can be made much smaller than the distance between $\sqrt{\alpha}$ and x, and going to the recursive definition of x_{n+1} this implies that x_{n+1} is sufficiently far from x. The limit argument says that $x \neq 0$ and is bounded away from infinity. So since $x_{n+1} = (x_n + \alpha/x_n)/2$ we can take the limit of both sides and since the x_n and x_{n+1} both converge to x, we find that x satisfies a quadratic equation with solutions $\pm\sqrt{\alpha}$. Thus since x is positive, $x = \sqrt{\alpha}$.

$\square$

(3) Problem 20 on page 8 of Rudin.

Solution. Consider $\epsilon < 0$. We can find $N \in \mathbf{N}$ s.t. $d(p_n, p_m) < \epsilon/2$ for $m, n > N$. But we can also find $M > N$ s.t. $d(p, p_{n_m}) < \epsilon/2$. Now for any $n > n_m$, by the triangle inequality $d(p, p_n) \leq d(p, p_{n_m}) + d(p_{n_m}, p_n) < \epsilon$. $\square$

(4) Problem 21 on page 82 of Rudin.

Solution. Note that $\cap_1^\infty E_n$ can contain at most one point: For say $p, q \in \cap_1^\infty E_n$. Then, $p, q \in E_n$ for each n, so $d(p, q) \leq diam(E_n)$, so we must have $d(p, q) = 0$; thus $p = q$.

To show that $\cap_1^\infty E_n$ is non-empty, consider a sequence $\{x_n\}$, $x_n \in E_n$ for each n. Then, we note that for any $\epsilon > 0$ we can take N such that for $n > N$ we have $diam(E_n) < \epsilon$, so for $n, m > N$ we have $d(x_n, x_m) < diam(E_n) < \epsilon$ (for $x_n, x_m \in E_n$ as the E_i are nested). So, the $\{x_n\}$ form a Cauchy sequence which must converge to some $x \in X$. But then, x is a limit point of each E_n (by considering the tail of the sequence starting at x_n), and as each E_n is closed, it is a point of each E_n. Thus, $x \in \cap_1^\infty E_n$. $\square$

(5) Problem 22 on page 82 of Rudin.

Solution. I prove that the countable intersection is dense.

Let $O \subset X$ be any open set. Take some $x_0 \in O$ and r_0 such that $\bar{N}_{r_0}(x_0) \subset G$ (we can do this as G is open).

As G_1 is dense in X and $N_{r_0}(x_0)$ is open, we can take some $x_1 \in G_1 \cap N_{r_0}(x_0)$. $G_1 \cap N_{r_0}(x_0)$ is the intersection of open sets, thus open, and non-empty for it contains x_1, so we can take $r_1 < \frac{r_0}{2}$ so that $N_{r_1}(x_1) \subset \bar{N}_{r_1}(x_1) \subset N_{r_0}(x_0) \cap G_1$.

As G_2 is dense in X, we can choose some $x_2 \in G_2 \cap N_{r_1}(x_1)$; again, $G_2 \cap N_{r_1}(x_1)$ is a non-empty open set, so we can take $r_2 < \frac{r_1}{2}$, and then $N_{r_2}(x_2) \subset \bar{N}_{r_2}(x_2) \subset N_{r_1}(x_1) \cap G_2$.

Repeating this construction, we can always pick $x_{n+1} \in G_{n+1} \cap N_{r_n}(x_n)$, with $r_{n+1} < \frac{r_n}{2}$ and so $N_{r_{n+1}}(x_{n+1}) \subset \bar{N}_{r_{n+1}}(x_{n+1}) \subset G_{n+1} \cap N_{r_n}(x_n)$.

Then, $d(x_j, x_k) < r_j \left(\frac{1}{2} + \ldots + \frac{1}{2^{k-j}} \right) < r_j \leq \frac{r_0}{2^j}$, so the $\{x_k\}$ form a Cauchy sequence, and must have a limit point, $x \in X$. Now, we have $x \in \bar{N}_{r_n}(x_0)$, as each of these is closed and x is a limit point of a sequence in each. But then, for all $n \geq 1$ we have $x \in G_n$, as $\bar{N}_{r_n} x_n \subset G_n$, so $x \in S$. Also, we have that $x \in \bar{N}_{r_0}(x_0) \subset O$, so $x \in O \cap S$.

As O was arbitrary, we have that S is dense in X. $\qquad\square$

3 The Math Puzzler - just for fun!

Each week there will be a "math puzzler" for the class to think about. Please feel free to submit a "puzzler" you think the class might enjoy. The "puzzlers" don't have to be difficult, nor related to the material in class — they just have to be fun to think about!

This problem is from Ivan Corwin.

Given a 5×5 grid of dots, can you draw a closed curve, only using horizontal or vertical lines, and not lifting your pencil, such that every dot is crossed once and such that you end at your starting point? Can you generalize this problem? That is, for what $m \times n$ is this impossible?

Now consider a $5 \times 5 \times 5$ array of dots with the same rules (only go up, down, left right, in, out and go through every dot only once and end at the starting point). Is is possible to draw a closed curve with these rules? For what size three dimensional arrays is it impossible? Generalize to all dimensions.

Solution. Think Pick's Theorem. $\qquad\square$