Math 25a Homework 6 Solutions

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1 Ivan's problems

(1) Problem 14 on page 80 of Rudin.

Solution.

(a) Consider the sequence $x_n = s_n - s$, and $\phi_n = \sigma_n - s = \frac{x_0+x_1+\ldots+x_n}{n+1}$.

Then, $\lim s_n = s$ implies $\lim_{n\to\infty} x_n = 0$. So, for any $\epsilon > 0$, we may take N such that $n > N$ implies $|x_n| < \frac{\epsilon}{2}$. Then, $|\phi_n| = \frac{|x_1+\ldots+X_n+\ldots+x_n|}{n+1} < \frac{|x_1+x_2+\ldots+x_N|}{n+1} + \frac{\epsilon}{2}$. Now, take $M > \frac{2\epsilon}{\max(|x_1+x_2+\ldots+x_N|,1)} - 1$. So, for $n > \max N, M$, the previous quantity yields $|\phi_n| < \epsilon$. So, $\lim_{n\to\infty} \phi_n = 0$. This yields our desired result.

(b) Let $s_k = (-1)^k$. It is clear that s_k does not converge. However, $\sigma_{2k} = \frac{1}{2k+1}$ and $\sigma_{2k+1} = 0$, so $\lim_{k\to\infty} \sigma_k = 0$.

(c) Let $s_{2^n} = n$ for all n and $s_k = 2^{-k}$ for all k not a power of 2. Note that $\limsup_{n\to\infty} s_n = +\infty$, due to the power-of-two terms. For any $n \in \mathbf{N}$, we can uniquely write $n = 2^k + m$, with $0 \le m < 2^{k+1} - 2^k$. Then, $\sigma_n = \sigma_{2^k+m} \le \frac{1}{2^k+m+1}\left(1 + 2 + \ldots + k + \sum_{j=0}^{\infty} \frac{1}{2^j}\right) \le \frac{k(k+1)/2+2}{2^k+m+1}$. As $n \to \infty$, $k \to \infty$, so $\sigma_n \to 0$. So, our given sequence satisfies these conditions.

(d) We first show the identity given. Note that:

$$\sum_{k=1}^{n} ka_k = \sum_{k=1}^{n}\sum_{j=1}^{k} a_k = \sum_{j=1}^{n}\sum_{k=j}^{n} a_k$$

Telescoping:

$$= \sum_{j=1}^{n} s_n - s_{j-1} = s_n(n) - (n)\sigma_{n-1} = s_n(n+1) - (n+1)\sigma n$$

Then, dividing by $n+1$ on both sides we get our desired identity:

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} ka_k$$

Now, we know that $\lim_{n\to\infty}(na_n) = 0$ so we can take N, s.t. for $n > N$ we have $|na_n| < \frac{\epsilon}{2}$.

Denoting $A = \sum_{k=1}^{N} |ka_k| - \frac{N\epsilon}{2}$: Take $m > \max(N, \frac{2A}{\epsilon} - 2)$, and then we have:

$$
\begin{aligned}
|s_m - \sigma_m| &= \frac{1}{m+1}\left|\sum_{k=1}^{m} ka_k\right| \le \frac{1}{m+1}\sum_{k=1}^{m}|ka_k| \\
&= \left(\frac{1}{m+1}\right)\left(\sum_{k=1}^{N}|ka_k| + \sum_{k=N+1}^{m}|ka_k|\right) <= \left(\frac{1}{m+1}\right)\left(\sum_{k=1}^{N}|ka_k| + (m-N)\frac{\epsilon}{2}\right) \\
&= \left(\frac{1}{m+1}\right)\left[\left(\sum_{k=1}^{N}|ka_k| - \frac{N\epsilon}{2}\right) + \frac{m\epsilon}{2}\right] \\
&= \frac{m\frac{\epsilon}{2} + A}{m+1} = \frac{\epsilon}{2} + \frac{A - \frac{\epsilon}{2}}{m+1} < \epsilon
\end{aligned}
$$

Thus, $s_n - \sigma_n$ converges, and by the problem σ_n converges. So their termwise sum, s_n, converges.

(e) If $m < n$, then:

$$
\begin{aligned}
\left(\frac{m+1}{n-m}\right)(\sigma_n - \sigma_m) &= \left(\frac{m+1}{n-m}\right)\left(\frac{\sum_{k=0}^{m} s_k}{n+1} - \frac{\sum_{k=0}^{m} s_k}{m+1}\right) \\
&= \left(\frac{m+1}{n-m}\right)\left(\frac{(m-n)\sum_{k=0}^{m} s_k + (m+1)\sum_{k=m+1}^{n} s_k}{(m+1)(n+1)}\right) \\
&= -\frac{\sum_{k=0}^{m} s_k}{n+1} + \frac{m+1}{(n+1)(n-m)}\sum_{k=m+1}^{n} s_k
\end{aligned}
$$

Also,

$$
\frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i) = \frac{1}{n-m}\left(s_n(n-m) - \sum_{k=m+1}^{n} s_k\right) = s_n - \frac{1}{n-m}\sum_{k=m+1}^{n} s_k
$$

Then, the sum of these quantities is $s_n - \frac{\sum_{k=0}^{m} s_k}{n+1} - \frac{\sum_{k=m+1}^{n} s_k}{n+1} = s_n - \sigma_n$. Then, for these i in the summation,

$$
|s_n - s_i| = |a_n + a_{n-1} + \ldots + a_{i+1}| \le \frac{(n-i)M}{i+1} \le \frac{(n-m-1)M}{m+2}
$$

Now, for any given $\epsilon > 0$, for each integer n there must be a unique integer m such that $m \le \frac{n-\epsilon}{1+\epsilon} < m+1$ (as any real lies between two consecutive integers). From the left part, $m + m\epsilon \le n - \epsilon$, so $m + m\epsilon + \epsilon - m \le n - \epsilon + \epsilon - m \Rightarrow \frac{m+1}{n-m} \le \frac{1}{\epsilon}$. From the right part, $n - \epsilon < m\epsilon + 1 + \epsilon + m \Rightarrow \frac{n-m-1}{m+2} < \epsilon$, so $|s_n - s_i| < M\epsilon$. Then, $\limsup_{n\to\infty}|s_n - \sigma| \le M\epsilon$. Since ϵ was arbitrary, let it go to zero, and we have $\lim_{n\to\infty} s_n = \sigma$. □

(2) Let S be the set of positive integers that do not involve the digit 0 in their decimal representation (so, for example, $17 \in S$ but $107 \notin S$). Show that

$$
\sum_{n\in S}\frac{1}{n}
$$

converges to a value less than 90.

Solution. Call X_n the set of n-digit numbers that do not contain 0. There are 9^n numbers in X_n (9 choices for each digit), and each number is at least 10^{n-1}, so each reciprocal is at most $1/10^{n-1}$. Summing this over all of n gives 90. $\square$

(3) Tom Coates told me this problem which originally came from Dick Gross, dean of Harvard College.

(a) Let

$$s(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Show that this power series converges for $x \in (-1, 1]$. Give an argument to show that

$$s'(x) = \frac{1}{1+x^2}$$

and hence that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \arctan(1) = \frac{\pi}{4}.$$

(Do not worry about making the latter part of your argument rigorous. We will deal with differentiation of power series etc. later in the course in Spring semester.)

(b) Let

$$a_1 = 1 \quad a_2 = -1/3 \quad a_3 = 1/5 \quad a_4 = -1/7 \quad \ldots$$

and consider the partial sums

$$s_1 = a_1 \quad s_2 = a_1 + a_2 \quad s_3 = a_1 + a_2 + a_3 \quad \ldots$$

Show that

$$s_2 < s_4 < s_6 < \cdots < \frac{\pi}{4} < \cdots < s_5 < s_3 < s_1$$

(c) Let

$$t_1 = \frac{s_1 + s_2}{2} \quad t_2 = \frac{s_2 + s_3}{2} \quad t_3 = \frac{s_3 + s_4}{2} \quad \ldots$$

Show that

$$t_2 < t_4 < t_6 < \cdots < \frac{\pi}{4} < \cdots < t_5 < t_3 < t_1$$

(d) Let

$$u_1 = \frac{t_1 + t_2}{2} \quad u_2 = \frac{t_2 + t_3}{2} \quad u_3 = \frac{t_3 + t_4}{2} \quad \ldots$$

Show that

$$u_2 < u_4 < u_6 < \cdots < \frac{\pi}{4} < \cdots < u_5 < u_3 < u_1$$

(e) Use part (d) and the first 10 terms of the sequence a_n to give an estimate for π.

Solution. (a) Write $s(x) = \Sigma(a_n x^n)$. $limsup(|a_n|)^{1/n} = lim(1/n)^(1/n) = 1$, so the radius of convergence is 1. Also, $s(1)$ satisfies Leibnitz's conditions for alternating series (Rudin pp. 71). Therefore, we have convergence on $(-1, 1]$. Also, assuming we can differentiate, $s'(x) = 1 - x^2 + x^4 - x^6 \ldots = 1/(1 + x^2)$ by noticing the geometric series. Checking the constant of integration (by, for example, $s(0)$), we get $s(x) = arctan(x)$ after integration.

(b) Note that $s_{2n+2} - s_{2n} = a_{2n+1} + a_{2n+2} = 1/(4n+3) - 1/(4n+5) > 0$. So we have $s_2 < s_4 < \ldots$. Similarly, we have $s_1 > s_3 > \ldots$. s_n goes to $\pi/4$, so s_{2n} also goes to $\pi/4$. But s_{2n} is a monotonic sequence, so it can never exceed $\pi/4$. Similar reasoning for s_{2n+1} completes the proof.

(c) Note that the limit of t_n is $lim 1/2(s_n + s_{n+1} = 1/2(lim s_n + lim s_{n+1}) = \pi/4$. Also, $t_{2n+2} - t_{2n} = 1/2(s_{2n+2} + s_{2n+3} - s_{2n} - s_{2n+1}) = 1/2(a_{2n+1} + a_{2n+3} - 2a_{2k+2}) > 0$ (by convexity - this is cheap, but whatever. Do the algebra). So as above, t_{2n} and t_{2n+1} are both monotonic and tends to $\pi/4$, giving our desired inequality.

(d) Note that the limit of u_n is $lim 1/2(t_n + t_{n+1} = 1/2(lim t_n + lim t_{n+1}) = \pi/4$. Also, $u_{2n+2} - u_{2n} = 1/2(t_{2n+2} + t_{2n+3} - t_{2n} - t_{2n+1}) = 1/4(s_{2n+1} + s_{2n+3} - 2s_{2k+2}) > 0$ (by convexity - this is cheap, but whatever. Do the algebra). So as above, u_{2n} and u_{2n+1} are both monotonic and tends to $\pi/4$, giving our desired inequality.

(e) You should get something like $3.1409 < \pi < 3.1426$.

□