Math 25a Homework 8 Solutions

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1 Ivan's problems

(1) Problem 15 on page 36 of Axler.

Solution. A simple counterexample: Let $V = \mathbb{R}^2$ and $U_1 = \{(x, 0) : x \in \mathbb{R}\}$, $U_2 = \{(0, y) : y \in \mathbb{R}\}$ and $U_3 = \{(x, x) : x \in \mathbb{R}\}$. Then $U_1 + U_2 + U_3 = \mathbb{R}^2$, so $dim(U_1 + U_2 + U_3) = 2$. However, $dim(U_i) = 1$ for $i = 1, 2, 3$, so our formula would reduce to $2 = 3$ which is clearly false. □

(2) Problem 17 on page 36 of Axler.

Solution. Suppose that $U_1, \ldots, U_m$ are subspaces of V such that $V = U_1 \oplus \cdots \oplus U_m$. For each $j = 1, \ldots, m$, choose a basis for U_j. Put these bases together to form a single list B of vectors in V. Clearly B spans $U_1 + \cdots + U_m$, which equals V. If we show that B is also linearly independent, then it will be a basis of V. Thus the dimension of V will equal the number of vectors in B and hence the desired results.

We still need to show that B is linearly independent. To do this, suppose that some linear combination of B equals 0. Write this linear combination as $u_1 + \cdots + u_m$ where we have grouped together the terms that come from the basis vectors of U_i and called them u_i. Thus $u_1 + \cdots + u_m = 0$. However this implies that each $u_i = 0$ which from linear independence of the U_i implies that each coefficient in the original linear combination of the vectors in B was 0. Hence we have linear independence.

□

(3) Let V be a vector space over the field F and let W be a subspace of V. Define a relation $\sim$ in V by: $v_1 \sim v_2$ if and only if $v_1 - v_2 \in W$.

(a) Prove that $\sim$ is an equivalence relation.

The set of all equivalence classes is denoted V/W — the quotient space of V with respect to W. Let $[v_1]$ denote the equivalence class of v_1. Now we show V/W is also a vector space in the following way:

(b) Define addition on V/W and show that it is well defined. That is $[v_1]+[v_2] =?$ and show that it is independent of the representatives you have chosen.

(c) Define scalar multiplication and show that it is well defined. That is $a\cdot [v_1] =?$ (where $a \in F$), and show that it is independent of the representatives you have chosen.

(d) Now choose **two** vector space axioms and check that they hold. Submit your proofs. Convince yourself that all the axioms hold as well (but don't worry about submitting these extra proofs).

Solution. (a) Clearly $v-v = 0 \in W$ by additive closure. Similarly if $v_1-v_2 \in W$, multiplying by -1 gives that $v_2 - v_1 \in W$. Finally if $v_1 - v_2 \in W$ and $v_2 - v_3 \in W$, closure under addition implies $v_1 - v_3 \in W$ as desired.

(b) We define $[v_1] + [v_2] = [v_1 + v_2]$. To see that this is well-defined, assume we have v_1' and v_2' related, respectively, to v_1 and v_2. Then it suffices to show that $v_1 + v_2 \sim v_1' + v_2'$. However, since $v_1 - v_1' \in W$ and $v_2 - v_2' \in W$, addition implies $(v_1 + v_2) - (v_1' + v_2') \in W$ as desired.

(c) We write $a \cdot [v_1] = [av_1]$. If we have $v_1' \sim v_1$ then $v_1 - v_1' \in W$ implies $av_1 - av_1' \in W$ so $[av_1] = [av_1']$ as desired.

(d) We will do commutativity and associativity. Consider

$$[v_1] + [v_2] = [v_1 + v_2] = [v_2 + v_1] = [v_2] + [v_1].$$

The consider

$$[v_1]+([v_2]+[v_3]) = [v_1]+[v_2+v_3] = [v_1+v_2+v_3] = [v_1+v_2]+[v_3] = ([v_1]+[v_2])+[v_3].$$

□

(4) Suppose that for each $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, the function f is defined by:

(a) $f(x) = x_1 + x_2$,
(b) $f(x) = x_1 - (x_2)^2$,
(c) $f(x) = x_1 + 1$,
(d) $f(x) = 2x_2 - 3x_3$.

In which of these cases is f a linear functional?

Solution. (a) Clearly linear

(b) Not linear due to the squaring

(c) Not linear because of affine translation (adding 1)

(d) Clearly linear □

(5) Supose that for each x in $\mathcal{P}(\mathbb{R})$ (polynomials with real coefficients, in say the variable t), the function f is defined by:

(a) $f(x) = \int_{-1}^{+2} x(t)dt$,

(b) $f(x) = \int_{0}^{2} (x(t))^2 dt$,

(c) $f(x) = \frac{dx}{dt}$,

(d) $f(x) = \frac{dx}{dt}|_{t=1}$.

In which of these cases is f a linear functional?

Solution. (a) Maps to field and is linear from integral, so it is a linear functional.

(b) Is not linear due to square, so NO

(c) Does not map to the field, so NO

(d) Maps to field and is linear from derivative, so it is a linear functional. □