Math 25b Homework 9 Solutions Part 1

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As usual, solutions only to selected problems.

1 Alison's problems

(1) *Volumes and Determinants*
Problem 3–35 on page 62 of Spivak.

Solution. People didn't have much trouble with part a). However, as a few of you noticed, deducing part b) from part a) is actually trickier than it looks. The catch is that in order to prove part b), you need to use the fact that for g a linear transformation of the sort in part a), $v(g(U)) = \det gv(U)$ in situtations where U may not be a rectangle, but could be a general parallelogram. There is more than one way to deal with this: what we'll do is first use part a) to deduce a stronger version of a) where U can be any Jordan-measurable shape, not just a rectangle.

So let U be any Jordan-measurable shape, and g be any elementary linear transformation (i.e. of the type in part a). We claim that $v(g(U)) = \det gv(U)$. By Spivak's definition, $v(U) = \int_R \chi(U)$, where R is some rectangle containing U, and $\chi(U)$ is the characteristic function of U, which has the value 1 on U and 0 elsewhere. If you think about what this means in terms of Riemann sums, an equivalent definition is that $v(U)$ is the infimum of the total areas of all collections of rectangles intersecting each other on their boundaries containing U, and the supremum of the total areas of all collections of rectangles intersecting each other on their boundaries contained in U. That is, for any $\epsilon > U$, we can find a finite number of rectangles intersecting each other on their boundaries $R_1, R_2, \ldots, R_n$ covering U of total area $< v(U) + \epsilon$, and a finite number of rectangles intersecting each other on their boundaries $S_1, S_2, \ldots, S_m$ contained in U of total area $> v(U) - \epsilon$. We know from a) that applying g to a rectangle multiplies its area by $\det g$. So $g(R_1), g(R_2), \ldots, g(R_m)$ cover $g(U)$, intersect each other only on their boundaries, and have total area $\det g(v(U) + \epsilon)$: so $g(U)$ is contained in a region of area $\det g(v(U) + \epsilon)$, which means that $v(g(U)) \leq (\det g)(v(U) + \epsilon)$. Similarly, $v(g(U)) \geq (\det g)(v(U) - \epsilon)$. But we can make ϵ arbitrarily small, so $v(g(U)) = \det g(v(U))$ as desired.

Now we can use the argument that you all tried to make on your problem sets. First of all, from the linear algebra we did in Math 25a, any linear transformation g can be written as $g_1 g_2 \cdots g_n$, where $g_1, \ldots, g_n$ are elementary transformations as in part a). (If you think about it, you'll see this can be done even if g is not invertible, by letting some of the factors by elementary transformations of the first type with $a = 0$, i.e., that send e_k to 0 and preserve all other basis vectors.) Then, applying the result of the previous paragraph,

$$v(g_1 g_2 \cdots g_n(U)) = (\det g_1) v(g_2 \cdots g_n(U)) = \cdots = (\det g_1)(\det g_2) \cdots (\det g_n) v(U) = (\det g)(U).$$

using the fact that the determinant is multiplicative. The point is that the shapes we get in the intermediate stages, such as $g_2 \cdots g_n(U)$, may not be rectangles any more, but we can use the more general result proved above to deal with them.

(2) You all did well, I don't really have anything to say about this.

(3) People also did well on this problem; the only tricky parts were the integrals with x on the inside. The region we're integrating over is the set $\{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x^2\}$. In some sense, y and z are independent of each other here: y and z can take on any values in $[0, 1]$ independently of each other (i.e. if $x = 0$), but once you've chosen values for y and z, our conditions tell us that $x \leq 1 - y$, and $x \leq \sqrt{1-z}$, that is, $x \leq \min(1 - y, \sqrt{1-z})$, so x is constrained to the interval $[0, \min(1 - y, \sqrt{1-z}]$. This means that one way we can write our integral is as

$$\int_0^1 \int_0^1 \int_0^{\min(1-y,\sqrt{1-z})} 1dxdydz \text{ and } \int_0^1 \int_0^1 \int_0^{\min(1-y,\sqrt{1-z})} 1dxdzdy.$$

If we don't want to write things in terms of mins, we can break the integral up instead. We know that $\min(1 - y, \sqrt{1-z})$ equals $1 - y$ provided that $1 - y \leq \sqrt{1-z}$: equivalently, $y \geq 1 - \sqrt{1-z}$ or $z \leq 1 - (1 - y)^2$, and equals $\sqrt{1-z}$ when $y \leq 1 - \sqrt{1-z}$ or $z \geq 1 - (1 - y)^2$. So

$$\int_0^1 \int_0^1 \int_0^{\min(1-y,\sqrt{1-z})} 1dxdydz = \int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} 1dxdydz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} 1dxdydz$$

and likewise

$$\int_0^1 \int_0^1 \int_0^{\min(1-y,\sqrt{1-z})} 1dxdzdy = \int_0^1 \int_0^{1-(1-y)^2} \int_0^{1-y} 1dxdzdy + \int_0^1 \int_{1-(1-y)^2}^1 \int_0^{\sqrt{1-z}} 1dxdzdy.$$

(Of course you didn't have to explain it in so much detail... I'm hoping my explanation makes thing clearer, though.)

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(4) *Is this obvious to you? (Apparently I'm not a mathematician....)*

Parts (d) and (e) of problem 3–41 on pages 73—74 of Spivak.

Solution. People generally did well with the actual integrals: the only hard part was taking care of the limits. The nicest way to do that was to note that $B_r \subset C_r \subset B_{\sqrt{2}r}$ (draw a picture!), and $e^{-(x^2+y^2)}$ is everywhere positive, so

$$\int_{B_r} e^{-(x^2+y^2)}dxdy \leq \int_{C_r} e^{-(x^2+y^2)}dxdy \leq \int_{B_{\sqrt{2}r}} e^{-(x^2+y^2)}dxdy.$$

If we let $r \to \infty$, $\sqrt{2}r \to \infty$ as well, so $\lim_{r\to\infty} \int_{B_r} e^{-(x^2+y^2)}dxdy = \lim_{r\to\infty} \int_{B_{\sqrt{2}r}} e^{-(x^2+y^2)}dxdy$. By the squeeze lemma, then, $\lim_{r\to\infty} \int_{C_r} e^{-(x^2+y^2)}dxdy$ must have the same value as the two limits it is squeezed between, that is,

$$\int_{B_r} e^{-(x^2+y^2)}dxdy = \int_{C_r} e^{-(x^2+y^2)}dxdy$$

as desired.

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