Math 25b Homework 10

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1 Ivan's Problems

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1. (a) Cylindrical coordinates are a generalization of two-dimensional polar coordinates by superposing a height axis (i.e. (r, θ, z) in cylindrical coordinates corresponds to $(r\cos\theta, r\sin\theta, z)$ in rectangular coordinates). Spherical coordinates are a system of coordinates natural for describing positions on a sphere. If a point P is represented by the spherical coordinates (ρ, θ, ϕ), then ρ is the distance between P and the origin, ϕ is the polar angle between the z-axis and the line from the origin to P, and θ is the azimuthal angle between the positive x-axis and the line from the origin to the projection of P onto the xy-plane. P corresponds to $(\rho\cos\theta\sin\phi, \rho\sin\phi\sin\theta, \rho\cos\phi)$ in rectangular coordinates. Also, consider an example for each of these coordinate systems. In cylindrical coordinates, $r = 1$ is a cylinder of radius 1, and in spherical coordinates, $\rho = 1$ is a sphere of radius 1. Additionally, note that you can also map these systems back to rectilinear coordinates using inverse trig functions.

   (b) For cylindrical coordinates, define

$$g_1 : (r, \theta, z) \mapsto (r\cos\theta, r\sin\theta, z).$$

Then $|\det g_1'(r, \theta, z)| = \left|\det \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\right| = r$. Hence, $\int_{g_1(A)} f = \int_A (f \circ g_1)\pi^1$ ("$dx\,dy\,dz = r\,dz\,dr\,d\theta$").
Similarly, for spherical coordinates, define

$$g_2 : (\rho, \theta, \phi) \mapsto (\rho\cos\theta\sin\phi, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

Then $|\det g_2'(\rho,\theta,\phi)| =$

$$\left|\det\begin{pmatrix}\sin\phi\cos\theta & -\rho\sin\phi\cos\theta & \rho\cos\phi\cos\theta\\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta\\ \cos\phi & 0 & -\rho\sin\phi\end{pmatrix}\right| = \rho^2\sin\phi.$$

So $\int_{g_2(A)} f = \int_A (f\circ g_2)(\pi^1)^2\sin\pi^3$ (“$dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$”).

(c) In cylindrical coordinates, the paraboloid is $z=\frac{r^2}{4}$ and the sphere (top half) is $z=\sqrt{5-r^2}$. The projection of the solid onto the xy-plane is the disk $r\le 2$, and hence the volume of the solid is $\int_0^{2\pi}\int_0^2\int_{\frac{r^2}{4}}^{\sqrt{5-r^2}} r\,dz\,dr\,d\theta = \frac{2\pi}{3}(5\sqrt{5}-4)$.

(d) We need $\frac{1}{2}\int_0^{\pi}\int_0^{2\pi}\int_a^b \rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{2\pi}{5}(b^5-a^5)$.

2. Let $f:\mathbb{R}^n\to\mathbb{R}$. Suppose $D_{1,2}f$ and $D_{2,1}f$ are cts. and there exists $a=(a_1,\dots,a_n)\in\mathbb{R}^n$ s.t. $D_{1,2}f(a)-D_{2,1}f(a)>0$. Since $D_{1,2}f$ and $D_{2,1}f$ are cts., there is a rectangle $A=[a_1-\epsilon_1,a_1+\epsilon_1]\times\dots\times[a_n-\epsilon_n,a_n+\epsilon_n]$ s.t. $D_{1,2}f-D_{2,1}f>0$ on A. Hence $\int_A(D_{1,2}f-D_{2,1}f)>0$. But we have a contradiction since by Fubini's Theorem, $\int_A(D_{1,2}f-D_{2,1}f)=$

$$\int_{a_n-\epsilon_n}^{a_n+\epsilon_n}\cdots\int_{a_1-\epsilon_1}^{a_1+\epsilon_1}\int_{a_2-\epsilon_2}^{a_2+\epsilon_2} D_{1,2}f\,dx_2\,dx_1\dots dx_n$$
$$-\int_{a_n-\epsilon_n}^{a_n+\epsilon_n}\cdots\int_{a_2-\epsilon_2}^{a_2+\epsilon_2}\int_{a_1-\epsilon_1}^{a_1+\epsilon_1} D_{2,1}f\,dx_1\,dx_2\dots dx_n = 0.$$

3. (a) Let $f:(x,y)\mapsto(x+y)e^{x^2-y^2}$, let $R'=[0,\sqrt{2}]\times[0,\frac{3\sqrt{2}}{2}]$, and let g be the linear transformation with matrix $\begin{pmatrix}\cos\frac{\pi}{4} & \sin\frac{\pi}{4}\\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix}$. Since $Dg=g$, we have $|\det g'|=1$. Hence $\iint_R f(x,y)\,dx\,dy$

$$= \iint_{R'} f\left(\frac{1}{\sqrt{2}}(y+x),\frac{1}{\sqrt{2}}(y-x)\right)dx\,dy$$
$$= \int_0^{\frac{3\sqrt{2}}{2}}\int_0^{\sqrt{2}}\sqrt{2}ye^{2xy}\,dx\,dy = \frac{1}{4}(e^6-7).$$

(b) Let $R=\{(x,y)\in\mathbb{R}^2 \mid \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$ and let g be the linear transformation with matrix $\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}$. Again since $Dg=g$, we have $|\det g'|=ab$. Then $\iint_R dA = \iint_{B_1(0)} ab\,dA = \pi ab$. So the area bounded by the ellipse $9x^2+4y^2=1$ is $\pi\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)=\frac{\pi}{6}$.

4. (a) Let $v_1 = (a_1, a_2, a_3, a_4)$, $v_2 = (b_1, b_2, b_3, b_4) \in \mathbb{R}^4$. Then we have $w(v_1, v_2) + w(v_2, v_1) = f^1(v_1)f^2(v_2) - f^2(v_1)f^1(v_2) = a_1b_2 - a_2b_1 + b_1a_2 - b_2a_1 = 0$, so $w \in \Omega^2(V)$.

   (b) We have $(w \otimes w)(e_1, e_2, e_1, e_2) = w(e_1, e_2)w(e_1, e_2) = 1$ but $(w \otimes w)(e_1, e_1, e_2, e_2) = 0$ so $w \otimes w \notin \Omega^2(V)$.