Math 25b Homework 11

May 2, 2006

1 Ivan's Problems

Special thanks to Yifei Chen for providing the tex for these solutions.

1. Let $v_1, \ldots, v_n$ be an orthonormal basis with respect to T, and define $A = (a_{ij})$ by $w_i = \sum_{j=1}^n a_{ij} v_j$. Then, $g_{ij} = T(w_i, w_j)$

$$= T\left(\sum_{k=1}^n a_{ik} v_k, \sum_{k=1}^n a_{jk} v_k\right) = \sum_{k,l=1}^n a_{ik} a_{jl} T(v_k, v_l) = \sum_{k=1}^n a_{ik} b_{jk}.$$

It follows that $\det(g_{ij}) = \det(A \cdot A^T) = (\det A)^2$, and hence by Spivak 4-6, $|\omega(w_1, \ldots, w_n)| = |(\det A)\omega(v_1, \ldots, v_n)| = |\det A| = \sqrt{\det(g_{ij})}$.

2. If $v_p \in \mathbb{R}^n_p$, then by the Chain Rule we have $(g_* \circ f_*)(v_p)$

$$\begin{aligned} &= g_*((Df(p)(v))_{f(p)}) = ((Dg(f(p)) \circ Df(p))(v))_{(g\circ f)(p)} \\ &= (D(g \circ f)(p)(v))_{(g\circ f)(p)} = (g \circ f)_*(v_p). \end{aligned}$$

Now, if ω is a k-form on $\mathbb{R}^p$ and $v_q = (v_q^1, \ldots, v_q^k) \in (\mathbb{R}^n_q)^k$, then we have $((f^* \circ g^*)\omega)(q)(v_q)$

$$\begin{aligned} &= (f^* \circ g^*)(\omega((g \circ f)(q)))(v_q) \\ &= \omega((g \circ f)(q))((g_* \circ f_*)(v_q^1), \ldots, (g_* \circ f_*)(v_q^k)) \\ &= \omega((g \circ f)(q))((g \circ f)_*(v_q^1), \ldots, (g \circ f)_*(v_q^k)) \\ &= (g \circ f)^*(\omega((g \circ f)(q)))(v_q) = ((g \circ f)^*\omega)(q)(v_q). \end{aligned}$$

People who lost points here forgot what a pullback is applied to. For every point of our space, we apply the pullback to alternating forms on the tangent space at the point. Hence a pullback of a k-form must be applied to k vectors in the tangent space at the point p.

3. (a) The tangent vector $(f \circ c)_*((e_1)_t) = (f_* \circ c_*)((e_1)_t) = f_*(v)$.
   (b) Since $|c(t)| = 1$ for all t, we have $\sum_{k=1}^n c^k(t)^2 \equiv 1$, and hence $\sum_{k=1}^n 2c^k(t)(c^k)'(t) = 2\langle c(t)_{c(t)}, c_*((e_1)_t)\rangle_{c(t)} \equiv 0$.

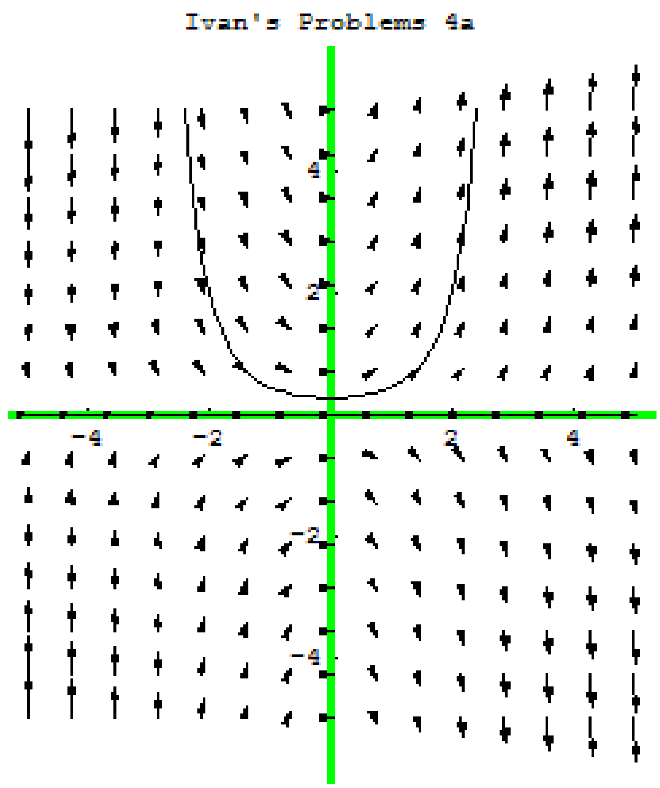

Figure 1: Notice that the field has mirror symmetry. Many people messed up the drawing of this field and hence of the flow lines for the negative x areas.

4. (a)
   (b) Suppose that γ is a flowline of F through (x, y) so that $\gamma(0) = (x, y)$. Then for all t, we have

$$((\gamma^1)'(t), (\gamma^2)'(t))_{\gamma(t)} = F(\gamma^1(t), \gamma^2(t)) = (1, \gamma^1(t)\gamma^2(t))_{\gamma(t)}.$$

   Thus, $(\gamma^1)'(t) \equiv 1$ so that $\gamma^1(t) \equiv t + x$ and $(\gamma^2)'(t) \equiv (t+x)\gamma^2(t)$ so that $\gamma^2(t) \equiv ye^{\frac{t^2}{2}+xt}$. Hence $\gamma(t) \equiv (t+1, 0), (t-4, 0)$, and $(t+2, 2e^{\frac{t^2}{2}+2t})$ respectively.

5. (a) We need $\nabla^2 f(x, y) = (f_{11} + f_{22})(x, y) = 2a + 2c \equiv 0$. Hence, if $a + c = 0$, then f is harmonic.

(b) If $x \in U$, then $f_{kk}(x_1, \ldots, x_n)$

$$\begin{aligned} &= \frac{\partial}{\partial x_k}\left((2-n)(x_1^2+\ldots+x_n^2)^{-\frac{n}{2}}x_k\right) \\ &= (n-2)(x_1^2+\ldots+x_n^2)^{-\frac{n}{2}-1}(nx_k^2-(x_1^2+\ldots+x_n^2)). \end{aligned}$$

Hence $\sum_{k=1}^n f_{kk}(x_1, \ldots, x_n)$

$$= (n-2)(x_1^2+\ldots+x_n^2)^{-\frac{n}{2}-1}\sum_{k=1}^{n}(nx_k^2-(x_1^2+\ldots+x_n^2)) = 0.$$

(c) Let $u = e^x \cos y$, $v = e^x \sin y$. Then $\frac{\partial u}{\partial x} = u$, $\frac{\partial u}{\partial y} = -v$, $\frac{\partial v}{\partial x} = v$, $\frac{\partial v}{\partial y} = u$. Thus, $f_{11}(x, y) = \frac{\partial}{\partial x}(ug_1(u, v) + vg_2(u, v)) = ug_1(u, v) + u(ug_{11}(u, v)+vg_{12}(u, v))+vg_2(u, v)+v(ug_{21}(u, v)+vg_{22}(u, v))$ and similarly $f_{22}(x, y) = -ug_1(u, v) - v(-vg_{11}(u, v) + ug_{12}(u, v)) - vg_2(u, v) + u(-vg_{21}(u, v) + ug_{22}(u, v))$. Hence, $(f_{11} + f_{22})(x, y) = (u^2 + v^2)(g_{11}(u, v) + g_{22}(u, v)) \equiv 0$ since g is harmonic.