Math 25b Homework 12

Due Friday 5th May 2006.
ABSOLUTELY NO EXTENSIONS ALLOWED ON THIS HOMEWORK!

Half of this problem set will be graded by Alison and half by Ivan. Please turn in problems from Section 1 separately from the problems in Section 2. Remember to staple each bundle of solutions and also to put your name on each!

1 Alison's problems

(1) *Filling in details of proofs about d*

(a) Problem 4-13 part (b) on page 96 of Spivak. (Note: you can't use Theorem 4–10 (2). This question asks you to fill in the missing part of the proof of that result.)

Solution: Let $f, g : \mathbb{R}^n \to \mathbb{R}$. We want to show $d(fg) = (df)f + f(dg)$. Now $d(fg) = \sum_i D_i(fg)dx^i = \sum_i [(D_i f)g + f(D_i g)]dx^i = (\sum D_i f dx^i)g + f(\sum D_i g dx^i) = dfg + fdg$.

(b) Fill in the detail missing in the proof of Theorem 4–10 (4). That is if ω is a 0-form and $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then $f^*(d\omega) = d(f^*\omega)$.

Solution: Remember $\omega : \mathbb{R}^m \to \mathbb{R}$ is a 0-form (that is a smooth function). Here we have a 0-form ω at $f(p)$ that is made into a 1-form $d\omega$ at $f(p)$ and then is pulled-back to p. It "eats" a vector at p which we'll denote v_p. (A similar line of reasoning hold for the other side of the equation.)
Thus $f^*(d\omega)(p)(v_p) = d\omega(f(p))(f_*(v_p))_{f(p)} = D\omega(f(p))(Df(p)(v_p))_{f(p)}$. We can write this as $f^*(d\omega)(p) = D\omega(f(p))Df(p)$. Now $f^*\omega(p) = \omega(f(p)) = \omega \circ f(p)$. Thus $d(f^*\omega(p)) = D(\omega \circ f)(p) = D\omega(f(p))Df(p)$.
Alison has suggested the following solution:
The key step in this problem is the chain rule. Because ω is a 0-form, $f^*(d\omega)$ and $d(f^*\omega)$ are both 1-forms, so we have to look at what they do to a tangent vector v based at a given point p (some people failed to notice this). Applying d to the 0-form $f^*\omega$ is equivalent to taking the total derivative of the function $f^*\omega = \omega \circ f$:

$$d(f^*\omega)(p)(v_p) = D(f^*\omega)|_p(v) = D(\omega \circ f)|_p(v).$$

We now apply the chain rule to $\omega \circ f$: $D(\omega \circ f)_p(v) = D(\omega)_{f(p)} Df_p(v)$. (This is the critical step of the proof – most of the people who missed this problem didn't get this step. When working with these things, you have to make sure to watch your parentheses, and keep track of which function you're differentiating and where you're evaluating it.)

On the other hand, by definition of the pullback,

$$f^*(d\omega)(p)(v_p) = (d\omega \circ f)(p) f^*(v_p) = d\omega(f(p))(Df(v))_{f(p)}.$$

We now convert from lowercase to uppercase D again: $d\omega(f(p))(Df_p(v))_{f(p)} = (D\omega)_{f(p)}(Df_p(v))$. This is the same as what we got from $d(f^*\omega)$, so we're done.

There's an alternate solution that writes everythig out in terms of $dx_1, \ldots, dx_n$: it comes to the same thing, only a bit messier.

(2) *How does d relate to div, grad and curl?*

(a) Show that the following diagram is commutative:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \text{0-forms} & \xrightarrow{d} & \text{1-forms} & \xrightarrow{d} & \text{2-forms} & \xrightarrow{d} & \text{3-forms} & \longrightarrow & 0 \\
\uparrow & & \uparrow R & & \uparrow S & & \uparrow T & & \uparrow U & & \uparrow \\
0 & \longrightarrow & \text{functions} & \xrightarrow{grad} & \text{vector fields} & \xrightarrow{curl} & \text{vector fields} & \xrightarrow{div} & \text{functions} & \longrightarrow & 0
\end{array}$$

(So this means you need to show that each square commutes. For example, $S \circ grad = d \circ R$.) In this diagram all vector fields, differential forms, and functions are on $\mathbb{R}^3$ and

$$R(f) = f$$

$$S\begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix} = F_1 dx + F_2 dy + F_3 dz$$

$$T\begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix} = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$$

$$U(f) = f dx \wedge dy \wedge dz$$

Solution: This is a straightforward calculation. The two outermost squares are trivial. I'll do one of the others. $S(\nabla(f)) = S(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df = d(R(f))$.

(b) Deduce that

$$curl(grad(f)) = 0$$

for all functions $f : \mathbb{R}^3 \to \mathbb{R}$ and that

$$div(curl\mathbf{F}) = 0$$

for all vector fields $\mathbf{F}$ on $\mathbb{R}^3$.

Solution: As the diagram commutes we conclude $T(curl(grad(f)) = d(d(R(f)) = 0$. Hence $curl(grad(f) = 0$. Similar reasoning gives $div(curl(F)) = 0$.

(3) *A closed form that is not exact*

Consider the 1-form defined on $A = \mathbb{R}^2 \setminus \{(0,0)\}$:

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

(a) Is ω closed? In other words, does $d\omega = 0$?

Solution: A short calculation shows $\omega = 0$.

(b) Let C_0 be the unit circle in $\mathbb{R}^2$ parameterized by the function $\gamma_0 : [0,1] \to \mathbb{R}^2$, where $\gamma_0(t) = (\cos 2\pi t, \sin 2\pi t)$. Find $\int_{C_0} \omega$.

Solution: $\gamma_0^*(dx) = -2\pi \sin t\, dt$ and $\gamma_0^*(dy) = 2\pi \cos t\, dt$, hence $\gamma_0^*(\omega) = 2\pi dt$. Thus

$$\int_{C_0} \omega = \int_{[0,1]} \gamma_0^* \omega = \int_0^1 2\pi dt = 2\pi.$$

(c) Is ω exact? In other words, does there exist a function $f : A \to \mathbb{R}$ such that $df = \omega$?

Solution: No. Assume $\omega = df$. Then

$$\int_{C_0} \omega = \int_{\gamma_0} \frac{\partial f}{\partial x} dx = \int_{\partial \gamma_0} f = f(\gamma_0(1)) - f(\gamma_0(0)) = 0.$$

Part (b) shows that this integral is 2π, a contradiction.

Let's give another take on what is going on. Note that the calculation above shows that if $\omega = df$ then $\int_\gamma \omega$ is entirely dependent on the value of f at the endpoints of the path γ. This is also equal to the change in angle between the two points. Now you might guess $f = \arctan(\frac{y}{x})$, which makes sense so long as $x \neq 0$. We can extend $\arctan(\frac{y}{x})$ across the y-axis away from the origin if we think of it geometrically in terms of the angle in polar coordinates. (You can also argue as in Spivak page 93, although if you do so, you should make sure to fill in the gap that Spivak doesn't prove, but leaves to the reader as exercise 4-21.)The part (b) and the calculation above gives the desired contradiction.

(d) Is the restriction of ω to $B = \mathbb{R}^2 \setminus \{(x,0) : x \leq 0\}$ exact?

Solution: Yes, just define f to be one branch of $\arctan(y/x)$ and check that $df = \omega$. You then have to fudge this a little by adding the proper multiples of π to make sure that this is continuous everywhere except on the negative y-axis – I'll leave the details to you.

For more on closed and exact forms see Ivan's problems (3) and (4).

(4) *Line Integrals*

(a) Compute the work done by a force field

$$\mathbf{F}(x, y, z) = \begin{pmatrix} y^2 \cos z \\ 2xy \cos z \\ -xy^2 \sin z \end{pmatrix}$$

moving a particle from $(2, 0, 0)$ to $(0, 0, 3)$ along the curve which is the intersection of the half ellipsoid

$$9x^2 + y^2 + 4z^2 = 36, \qquad z \geq 0$$

with the xz-plane.
(Hint: there is a fast way to do this question.)

Solution: Let $F = y^2 \cos z dx + 2xy \cos z dy - xy^2 \sin z dz$ be the form corresponding to $\mathbf{F}$. With thought you can see that $F = df$ for $f = xy^2 \cos z$. Let γ be the path (1-cube) of the particle. Then

$$\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_\gamma F = \int_\gamma df = \int_{\partial\gamma} f = f(\gamma(1)) - f(\gamma(0)) = f(0, 0, 3) - f(2, 0, 0) = 0.$$

Alternatively, note that $y = 0$ along the given path so $\mathbf{F} = 0$ along the path. (Hmmm, this problem was extra sneaky.)

(b) Suppose that $\mathbf{F}$ is a velocity field of a fluid moving in $\mathbb{R}^2$ such that one of the flowlines of $\mathbf{F}$ is a closed curve. Can $\mathbf{F}$ be conservative?

Solution: No. Let γ be the flow line. Then as above, $\int_\gamma \mathbf{F} \cdot d\mathbf{r} = \int_\gamma F$. As $\mathbb{R}^2$ is a star shaped region we can apply the fundamental theorem of line integrals. Thus $\mathbf{F}$ is conservative if and only if the integral $\int_\gamma \mathbf{F} \cdot d\mathbf{r}$ around any closed path is zero. In this case γ is a flow line. That is, every tangent vector to γ IS the field vector at that point. $F(\gamma(t)) \cdot \gamma'(t) = \|F(\gamma(t))\|^2 > 0$, and the integral $\int_{t=0}^1 F(\gamma(t)) \cdot \gamma'(t) dt > 0$ also, which contradicts the above. So $\mathbf{F}$ can't be conservative.

2 Ivan's problems

(1) *Gauss' Law*

The electric field $\mathbf{E}$ at a point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ not at the origin created by a point charge of charge Q at the origin is equal to

$$\frac{Q\epsilon x}{||x||^3}$$

where ϵ is a constant. (This is the electric force by a unit charge—a "test charge"—placed at x.)

(a) Write down the 2-form corresponding to the vector field $\mathbf{E}$. Is it closed?

Solution:

$$E = \frac{Q\epsilon x}{(x^2+y^2+z^2)^{3/2}} dy \wedge dz + \frac{Q\epsilon y}{(x^2+y^2+z^2)^{3/2}} dz \wedge dx + \frac{Q\epsilon z}{(x^2+y^2+z^2)^{3/2}} dx \wedge dy$$

After a big calculation you get terms like $\frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{5/2}} dx \wedge dy \wedge dz$. Everything cancels giving $dE = 0$.

(b) Compute the flux of **E** outwards through a sphere of radius $r > 0$ centered at the origin. (You should do this directly—calculate an appropriate integral.)

Solution: The sphere S is the image of the 2-cube given by $g(\phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi)$. (Think spherical coordinates.) Then $g^*(dx) = r \cos\phi \cos\theta d\phi - r \sin\phi \sin\theta d\theta$ and $g^*(dy) = r \cos\phi \sin\theta d\phi + r \sin\phi \cos\theta d\theta$ and $g^*(dz) = -r \sin\phi d\phi$. Hence $g^*(dx \wedge dy) = r^2 \cos\phi \sin\phi \, d\phi \wedge d\theta$ and $g^*(dy \wedge dz) = r^2 \sin^2\phi \cos\theta \, d\phi \wedge d\theta$ and $g^*(dz \wedge dx) = r^2 \sin^2\phi \sin\theta \, d\phi \wedge d\theta$. Some more algebra gives $g^* E = Q\epsilon \sin\phi \, d\phi \wedge d\theta$. Then

$$\int_S \mathbf{E} \cdot \hat{n} \, dA = \int_S E = \int_0^{2\pi} \int_0^{\pi} Q\epsilon \sin\phi \; d\phi d\theta = 4\pi Q\epsilon.$$

(c) Why do (a) and (b) not contradict Stokes's Theorem?

Solution: Note that E is not defined at the origin. Let $S = \partial B$. It is thus incorrect to say $\int_{\partial B} E = \int_B dE$. (Part (a) gives 0 for the second integral, part (b) gives the first integral nonzero—the apparent contradiction.)

(d) Explain why Stokes's Theorem implies *Gauss' Law*: The electric flux out of a smooth closed surface $S \subset \mathbb{R}^3$ is proportional to the charge enclosed by the surface.

Solution: This depends only on Q. Consider one point of charge at the origin. Enclose this by two spheres S_1 and S_2. Let S_2 have the larger radius. $S_1 \cup S_2$ bound a volume T and on this volume the net flux is zero (since $dE = div\mathbf{E} = 0$. Apply Stokes's theorem (in the form of the divergence theorem):

$$0 = \int_T dE = \iiint_T div\mathbf{E} \, dV = \iint_{\partial T} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1 \cup S_2} \mathbf{E} \cdot \mathbf{n} \, dS = -\int_{S_1} E + \int_{S_2} E.$$

Here we use the fact that the orientation of S_1 as part of ∂T is the opposite to the orientation of S_1 when considered as the boundary of the solid ball containing the origin. From part (b) we have $4\pi Q\epsilon = \int_{S_1} E = \int_{S_2} E$. This shows the electric flux of **E** is $4\pi Q\epsilon$ through any surface that contains the origin. A similar argument will hold for many charges. (Think swiss cheese. Enclose every charge Q_i in a small sphere S_i and let the new surface be a union of S with these spherical surfaces. This has net flux 0 by (a). But the flux coming in from the charges should be $\sum 4\pi Q\epsilon$ so the flux leaving is the same amount.)

(2) *Stokes's Theorem*

(a) Use Stokes's Theorem to evaluate

$$\int_C -y^2dx + xdy + z^2dz$$

where C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented anticlockwise when viewed from above.

Solution: Now $d(-y^2dx + xdy + z^2dz) = -2ydy \wedge dx + dx \wedge dy = (1 + 2y)dx \wedge dy = \omega$. The region D bounded by the curve C is parametrized by $\gamma(r, \theta) = (r\cos(2\pi\theta), r\sin(2\pi\theta), 2 - r\sin(2\pi\theta))$. Then $\gamma^*(dx) = \cos(2\pi\theta)dr - 2\pi r\sin(2\pi\theta)d\theta$ and $\gamma^*(dx) = \cos(2\pi\theta)dr - 2\pi r\sin(2\pi\theta)d\theta$. Hence $\gamma^*(\omega) = (1 + 2r\sin(2\pi\theta))2\pi r dr \wedge d\theta$.

$$\int_C -y^2dx + xdy + z^2dz = \int_0^1 \int_0^1 2\pi(r + 2r^2\sin(2\pi\theta))drd\theta = \pi.$$

(b) Let C be a smooth closed curve in $\mathbb{R}^2$, oriented anticlockwise. Show that

$$\int_C xdy = -\int_C ydx = \frac{1}{2}\int_C xdy - ydx = \text{the area bounded by } C.$$

Optional extra: What is the area of the polygon in $\mathbb{R}^2$ with vertices $(x_1, y_1), \ldots, (x_n, y_n)$?

Solution: Let R be the area bounded by C. Note that $d(xdy - ydx) = dx \wedge dy - dy \wedge dx = 2dx \wedge dy$. Hence $\int_{\partial R} xdy = -\int_{\partial R} ydx = \frac{1}{2}\int_{\partial R} xdy - ydx = \int_R dx \wedge dy$ This last integral is the area of R ($dx \wedge dy$ is the area form).

(c) This part is optional. **Eureka!** Problem 5–36 on page 137. For this problem "three manifold with boundary" just means a bounded region in $\mathbb{R}^3$ bounded by a smooth surface S.

Solution: Stokes's Theorem (in the guise of the Divergence Theorem) gives

$$-\int_{\partial M} \mathbf{F} \cdot \hat{n}\, dA = -\int_M div\mathbf{F}\, dV = -\int_M c\, dV.$$

Homology

We now follow up on Alison's Problem (3). A sequence of vector spaces and linear maps

$$\cdots \xrightarrow{T_{i-1}} V_i \xrightarrow{T_i} V_{i+1} \xrightarrow{T_{i+1}} V_{i+2} \xrightarrow{T_{i+2}} \cdots$$

is called a *complex* if $T_{i+1} \circ T_i = 0$ for all i. Exact sequences are examples of complexes. The sequence

$$0 \longrightarrow \Omega^0_{dR}(A) \xrightarrow{d} \Omega^1_{dR}(A) \xrightarrow{d} \Omega^2_{dR}(A) \xrightarrow{d} \Omega^3_{dR}(A) \longrightarrow 0$$

for an open set A of $\mathbb{R}^3$ is another example, called the *de Rham complex of A*. Given a complex, we define the *ith homology group* of the complex to be the quotient vector space

$$H_i = \ker T_i / \mathrm{im} T_{i-1}.$$

Note that this makes perfect sense, as $\mathrm{im}T_{i-1}$ is a subspace of $\ker T_i$.

The ith homology group of the de Rham complex of A is called the *ith de Rham cohomology group* of A, and is writen

$$H^i_{dr}(A).$$

Roughly speaking, it measures how many closed i-forms on A are not exact.

(3) *Zeroth cohomology*

Suppose A is a connected open subset of $\mathbb{R}^n$. Compute $H^0_{dR}(A)$. What happens if A is not connected?

Solution: $H^0_{dR} = \ker(d)/\mathrm{im}(0)$ where 0 is the linear map between 0 and $\Omega^0(A)$. As $\mathrm{im}(0) = 0$ (the 0-map), $H^0 = \ker d = \{f \mid df = 0\}$. Each such f is a locally constant function, that is constant on a connected set. Hence $H^0(A) \cong \mathbb{R}$ and if A has k-connected components, then $H^0(A) \cong \mathbb{R}^k$. (What is the explicit isomorphism?)

(4) *The Snake Lemma*

Suppose that we have a commutative diagram

$$\begin{array}{ccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & A_{k+1} & \xrightarrow{f_{k+1}} & B_{k+1} & \xrightarrow{g_{k+1}} & C_{k+1} & \longrightarrow & 0 \\
\uparrow & & \uparrow{\scriptstyle d_k^A} & & \uparrow{\scriptstyle d_k^B} & & \uparrow{\scriptstyle d_k^C} & & \uparrow \\
0 & \longrightarrow & A_k & \xrightarrow{f_k} & B_k & \xrightarrow{g_k} & C_k & \longrightarrow & 0 \\
\uparrow & & \uparrow{\scriptstyle d_{k-1}^A} & & \uparrow{\scriptstyle d_{k-1}^B} & & \uparrow{\scriptstyle d_{k-1}^C} & & \uparrow \\
0 & \longrightarrow & A_{k-1} & \xrightarrow{f_{k-1}} & B_{k-1} & \xrightarrow{g_{k-1}} & C_{k-1} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow
\end{array}$$

where the rows are exact and the columns are complexes. (So the diagram continues above and below what is shown.) Such an object is known as a *short exact sequence of complexes.*

(a) Show that the maps $f_* : H_k(A) \to H_k(B)$ where $[\omega] \mapsto [f_k(\omega)]$ and $g_* : H_k(B) \to H_k(C)$ where $[\eta] \mapsto [g_k(\eta)]$ are well defined.

Recall that the kth cohomology group is a quotient space $H_k(A) = \ker(d_k^A)/\mathrm{im}(d_{k-1}^A)$. Here

$\omega \in \ker(d_k^A)$ and $[\omega]$ is the equivalence class (or coset) $\omega + \mathrm{im}(d_{k-1}^A)$. A similar definition holds for $[\eta]$.

Solution: Throughout, I'll use the abbreviation CD for commuting diagram. We need to show that the definition does not depend on the representative of the equivalence class chosen. Here we have $[\omega] = \{\omega + d_{k-1}^A\phi | \phi \in A_{k-1}\}$. We'll prove this for f_*, the proof for g_* is similar. Now $\omega \in \ker d_k^A$, hence $d_k^A\omega = 0$. By CD $f_{k+1}d_k^A(\omega) = 0 = d_k^B f_k(\omega)$. Hence $f_k(\omega) \in \ker d_k^B$. Suppose $a \in \mathrm{im} d_{k-1}^A$, then $a = d_{k-1}^A\phi$, for $\phi \in A_{k-1}$. Then $f_k(\omega+a) = f_k(\omega)+f_k(d_{k-1}^A\phi) = f_k(\omega)+d_{k-1}^B(f_{k-1}\phi)$ (by CD). Hence $f_k(\mathrm{im} d_{k-1}^A) \subset \mathrm{im} d_{k-1}^B$ and so f_* is well defined.

(b) Construct a map $h_* : H_k(C) \to H_{k+1}(A)$ such that the following sequence is exact:

$$\ldots \longrightarrow H_k(A) \longrightarrow H_k(B) \longrightarrow H_k(C) \longrightarrow H_{k+1}(A) \longrightarrow H_{k+1}(B) \longrightarrow \ldots$$

In other words, *from a short exact sequence of complexes we can construct a long exact sequence of homology groups.* This will turn out to be very useful!

Having trouble getting started?

(i) Start with $\omega \in \ker(d_k^C)$ and try to construct an element $\nu \in \ker(d_{k+1}^A)$. In the end you want $[\nu] \in H_{k+1}(A)$ so you really want the image of your map to be well defined up to an ambiguity of the form $\nu \approx \nu + d_k^A(\phi)$. Now the map you want doesn't start at ω, but at $[\omega]$. So we need to check that if we change ω by an element of $\mathrm{im}(d_{k-1}^C)$ then $[\nu] \in H_{k+1}(A)$ doesn't change. (I can give you more of a hint as to how to go from ω to ν if you need it.)

(ii) Now you've defined h_* don't forget to check that the sequence is exact!

Solution: Throughout, I'll use the abbreviation CD for commuting diagram. We first define the map $h_*([\omega]) := [f_{k+1}^{-1} \circ d_k^B \circ g_k^{-1}(\omega)]$. (Note that this is the only definition that has a hope of making sense. Each row is exact, thus $\mathrm{im} f_k = \ker g_k$, hence $f_k^{-1}(g_k^{-1}(c))$ will only make sense if $c \in \ker g_k$.) So does our definition of h_* make sense? Yes. Given $[\omega]$, $\omega \in \ker d_k^C$ and $d_k^C\omega = 0$. As row is exact, g_k is onto. Let $b \in B_k$ such that $g_k(b) = \omega$. Then $d_k^C g_k(b) = g_{k+1}d_k^B(b)$ and so $d_k^B(b) \in \ker g_{k+1} = \mathrm{im} f_{k+1}$. Hence $f_{k+1}^{-1}(d_k^B(b))$ makes sense. Now what if we had picked a different $b' \in B_k$ such that $g_k(b') = g_k(b) = \omega$? For our map to be well-defined $f_{k+1}^{-1}d_k^B(b)$ and $f_{k+1}^{-1}d_k^B(b')$ must differ by something in $\mathrm{im} d_k^A$. By definition, b and b' differ by an element of $\ker g_k = \mathrm{im} f_k$ (by exactness). Thus $b' = b + f_k\alpha$ for $\alpha \in A_k$. Then $f_{k+1}^{-1}(d_k^B(b')) = f_{k+1}^{-1}(d_k^B(b)) + f_{k+1}^{-1}(d_k^B(f_k\alpha)) = f_{k+1}^{-1}(d_k^B(b)) + d_k^A\alpha$ by CD as desired. We need to check one more thing in order for h_* to be well defined. Namely suppose that $\omega' = \omega + d_{k-1}^C\gamma$ for $\gamma \in C_{k-1}$. Now $d_{k-1}^C(\gamma) = d_{k-1}^C(g_{k-1}(\beta))$ for $\beta \in B_{k-1}$ (as g_{k-1} is onto by exactness). But $d_k^B d_{k-1}^B(\beta) = 0$, that is $d_{k-1}^C(\gamma)$ maps to $\mathrm{im} d_k^A$ under h_*.

Now we have to show exactness of the sequence. That is (a) exactness at $H_k(B)$, namely $\mathrm{im} f_* = \ker g_*$ (b) exactness at $H_k(C)$, namely $\mathrm{im} g_* = \ker h_*$ (c) exactness at $H_{k+1}(A)$, namely $\mathrm{im} h_* = \ker f_*$.

(a) Exactness at $H_k(B)$. $[\omega] \in H_k(A)$, $f_*[\omega] = [f_k(\omega)] = [f_k(\omega) + f_k d_{k-1}^A(\phi)]$. Then $g_* f_*[(\omega]) = [g_k f_k(\omega) + g_k f_k d_{k-1}^A(\phi)] = [0 + d_{k-1}^C(g_{k-1}f_{k-1}(\phi))] = [0]$ (by exactness and

CD). Thus $\mathrm{im} f_* \in \ker g_*$. Now let $[\omega] \in \ker g_*$. We want $[a] \in H_k(A)$ and $f_*[a] = [\omega]$. By definition, $g_k(\omega) = 0 = d^C_{k-1}(\psi)$ where $\psi \in C_{k-1}$. Let $\phi \in B_{k-1}$ be such that $\psi = g_{k-1}(\phi)$. Now $g_k(\omega - d^B_{k-1}\phi) = g_k(\omega) - g_k d^B_{k-1}(\phi) = d^C_{k-1}(\psi) - d^B_{k-1} g_{k-1}(\phi) = d^C_{k-1}(\psi) - d^C_{k-1}(\psi) = 0$. Then $f_k(a) = \omega = d^B_{k-1}(\phi)$ for $a \in A_k$. By CD and exactness, $d^B_k f_k(a) = f_{k+1} d^A_k(a) = 0$, hence $d^A_k(a) = 0$ and $[a] \in H_k(A)$. Thus $\omega \in \mathrm{im} f_*$.

(b) Exactness at $H_k(C)$. For $\mathrm{im} g_* \in \ker h_*$ we want $h_* g_* = 0$. This follows from the definition of h_* and $\eta \in \ker d^B_k$, as $h_*([g_k(\eta) + g_k d^B_{k-1}\phi]) = h_*(g_k(\eta) + d^C_{k-1} g_{k-1}\phi) = [0+0] = [0]$. Now let $[c] \in \ker h_*$ using our definition of h_*, we see that there is a $b \in B_k$ such that $g_k(b) = c$ and there is an $a \in A_{k+1}$ such that $f_{k+1}(a) = d^B_k b$. Now $a \in \mathrm{im} d^A_k$ as $h_*(c) = 0$. Thus $a = d^A_k(a')$. Then $d^B_k f_k(a') = f_{k+1} d^A_k(a') = f_{k+1}(a) = d^B_k(b)$. Thus $d^B_k(b - f_k(a')) = 0$. Hence $g_k(b - f_k(a')) = g_k b - g_k f_k(a') = c - 0 = c$. Therefore $[g_k(b)] = [c]$.

(c) Exactness at $H_{k+1}(A)$. Similarly use the definition of h_* to deduce that $f_* h_* = 0$. Now let $[a] \in \ker f_*$. Then $f_{k+1}(a) = d^B_k(b)$ for $b \in B_k$. Let $c = g_k(b)$. Then $d^C_k(c) = d^C_k g_k(b) = g_{k+1} d^B_k(b) = g_{k+1} f_{k+1}(a) = 0$. Thus $[c]$ represents a cohomology class in $H_k(C)$ and by construction $h_*[c] = [a]$.

3 References and suggestions.

(1) Make sure you complete all the details in the worksheet from class on Friday 28th April. Also please work hard to understand the cohomology questions on this assignment. You can expect more questions on cohomology on the final exam that will build on this material!

(2) Additional references to Spivak:
a. The geometry of physics: an introduction by Theodore Frankel.
b. Geometrical methods of mathematical physics by Bernard Schutz. (Note: I currently have Cabot's copy out.)
c. Differential Topology by Guilleman and Pollack. Chapter 4 of this book goes over everything we are doing on tensors, forms and intergration, but over manifolds rather than $\mathbb{R}^n$.
d. Introduction to Differentiable Manifolds and Riemannian Geometry by William Boothby. (This is the text I'll use in Math 134 next year. It is very complete. The downside for you at the moment is the notation is slightly different to what we've been using, so you'll have to "translate" everything.)

(3) Additional references on (de Rham) cohomology:
a. Algebraic Topology: a first course by William Fulton.
b. Differential forms in Algebraic Topology by Bott and Tu—a classic text!
c. Math 135 is currently using a text called: From calculus to cohomology; de Rham cohomology and characteristic classes by Ib Madsen and Jorgen Tornehave. I haven't looked at

this book, but Professor Eftekhary has relayed the fact that it really does start at calculus and goes from there.