Math 25b Homework 1

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1 Ivan's problems

(1) Decide which of the matrices A are diagonalizable. If possible, find a diagonal matrix D and an invertible matrix S such that $S^{-1}AS = D$. (Do not use technology.)

$$A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \qquad A = \begin{pmatrix} 1/2 & 3/4 \\ 1/2 & 1/4 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution. People did fine on the first two, and people correctly identified the third as not being diagonalizable via looking at its eigenspaces. However, the fourth matrix is diagonalizable because it have a 2-dimensional zero eigenspace. The diagonal matrix ends up being 3 and two zeroes on the diagonal. □

We have the following metric spaces. (Check that they are indeed metric spaces but don't hand in!)

(a) $\mathbf{R}^n$, $d_1(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|$
(b) $\mathbf{R}^n$, $d_2(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$
(c) $\mathbf{R}^n$, $d_\infty(x,y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$
(d) $C([0,1])$, $d_1(f,g) = \int_0^1 |f(x) - g(x)|\, dx$
(e) $C([0,1])$, $d_\infty(f,g) = \max_x |f(x) - g(x)|$

(Here $C([0,1])$ is the set of continuous functions $f : [0,1] \to \mathbf{R}$. We will prove later that integrals and maxima always exist for such functions.)

(2) For the metrics d_1, d_2, d_∞ on $\mathbf{R}^2$, draw the open unit ball around 0. (This is denoted $B_1(0) = \{x \in \mathbf{R}^2 \mid d(x,0) < 1\}$.)

Solution. Everyone drew the correct shapes. Be careful in drawing that dotted lines means open and solid lines means closed. □

(3) Two metrics d_1 and d_2 on a set S are said to be *equivalent* if for any point $p \in S$, every open ball $B^1_R(p)$ with respect to the first metric contains an open ball $B^2_r(p)$ with respect to the second metric, and vice versa, every open ball $B^2_r(p)$ with respect to the second metric contains an open ball $B^1_R(p)$ with respect to the first metric.

(a) Prove that if two metrics d_1 and d_2 are equivalent on S, then a sequence $\{a_i\}$ in S converges with respect to d_1 if and only if it converges with respect to d_2.

(b) Now prove that on $\mathbf{R}^n$, the metrics d_1, d_2, d_∞ are all equivalent.

(c) Prove that the metrics d_1 and d_∞ on $C([0,1])$ are not equivalent by finding a sequence that converges with respect to d_1 but does not converge with respect to d_∞.

Solution. People understood these ideas well. The proofs in part b varied from explicit correspondences to scaling arguments. For part c you can take a function which starts at 1 and linearly drops to 0 in spacing $1/n$. These functions converge in d_1 to 0 but not in d_∞. $\square$