Math 25b Homework 2 Solutions Part 1

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1 Alison's problems

2 Alison's problems

(1) (a) Let $V = \mathbb{R}^n$ and recall from HW 1 that $M_n(\mathbb{R}) = \{A : V \to V \mid A \text{ is linear}\}$ and $GL_n(\mathbb{R}) = \{A : V \to V \mid A \text{ is invertible}\}$. Show that the closure of $GL_n(\mathbb{R}) \subset M_n(\mathbb{R})$ is all of $M_n(\mathbb{R})$.

(b) Let X be the collection of diagonalizable matrices in $M_2(\mathbb{R})$. Is X open, closed or neither? Explain.

Solution. (a) This is equivalent to showing that for any $A \in M_n(\mathbb{R})$ and any $\epsilon > 0$ there is a $B \in GL_n(\mathbb{R})$ with $d(A, B) < \epsilon$. For any $t \in \mathbb{R}$, let $A_t = A - tI$. Then A_t is a continuous function of t because it depends linearly on t in each coordinate. Furthermore, $\det(A_t) = \det(A - tI)$ is a polynomial in t – in fact, it is, up to sign, the characteristic polynomial of A, and the coefficent of t^n in $\det(A_t)$ is equal to ± 1. In particular, t^n is not the 0 polynomial, so it has only finitely many roots. By taking δ to be less than the smallest positive absolute value among the roots of $\det(A_t)$, then, we can guarantee that for $0 < |t| < \delta$, $\det A_t \neq 0$. Also, $t \mapsto A_t$ is continuous, so if we make δ sufficiently small we can also guarantee that $d(A_t - A) < \epsilon$ for $|t| < \delta$. Choose a δ satisfying both the above conditions. Then $A_{\delta/2} \in GL(n, \mathbb{R})$ with and $d(A_{\delta/2}, A) < \epsilon$. Because we can do this for any $\epsilon > 0$, A lies in the closure of $GL_n(\mathbb{R})$. But A could be anything in $M_n(\mathbb{R})$, so this closure is all of $M_n(\mathbb{R})$, and $GL_n(\mathbb{R})$ is dense in $M_n(\mathbb{R})$.

Comment: There were a number of ways to do this problem. The one I gave above is possibly the slickest, but a bunch of people did it by adding in small multiples of column vectors not spanned by the columns of the matrix to make the columns linearly independent.

(b) Neither. We first show that the point $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in X$ is not an interior point of X. First of all, I is diagonal, so it certainly lies in X. However, consider the matrix $\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$. This matrix is in Jordan block form but not diagonal, so it is not diagonalizable (this can also be checked explicitly. Furthermore, it is distance ϵ from A. We can make ϵ arbitrarily small, so any neighborhood of I contains a matrix of this form that does not lie in X. Hence I is not an interior point of X, so X is not open.

We now show that X is not closed either, by a similar method method. We claim that the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a limit point of X. This is a matrix with a non-diagonal Jordan block, so it is not diagonalizable and A is not in X. However, the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1+\epsilon \end{pmatrix}$ has two distinct eigenvalues given by its diagonal entries, so it is diagonalizable. As in the previous argument, we can make $d(A_\epsilon, A)$ arbitrarily small by making ϵ arbitrarily small, so any neighborhood of I contains an matrix in X. Hence A is a limit point of X that does not lie in X, so X is not closed. $\square$

(2) Problem 2 on page 98 of Rudin.

Solution. If $x \in E$ then the inclusion is clear. Otherwise, suppose x is a limit point of E. Then there is a sequence $\{x_n\}$ converging to x, $x_n \in E$. By f's continuity, $\{f(x_n)\}$ converges to $f(x)$. However, this means that $f(x)$ is a limit point of $f(E)$ and therefore in $\overline{f(E)}$. These two cases together prove the inclusion. To see that this needs not to be equivalence, take, say, $E = \mathbf{N}$ where $X = \mathbf{R}$, and $f(x) = 1/x$. Then $f(\overline{E}) = f(E)$ is the set of all reciprocals of natural numbers. But $\overline{f(E)}$ includes the limit points of these reciprocals and thus 0. $\square$

Comment: There is also a very nice proof that uses the fact that preimages of closed sets are closed.

(3) Problem 4 on page 98 of Rudin. (Please also read but not submit problems 5 and 13.)

Solution. Let U be a nonempty open set in $f(X)$. The preimage of U is open due to continuity, and is nonempty. But E is dense, so $f^{-1}(U) \cap E$ is nonempty, so $f(U) \cap f(E)$ is nonempty and $f(E)$ is dense. For the second part, take a sequence $\{p_n\}$ in E that converges to p, which exists due to density. Note that by continuity, $f(p) = f(limp_n) = limf(p_n) = limg(p_n) = g(limp_n) = g(p)$. $\square$

Comment: The first part of this problem is also a special case of problem 2, with $\bar{E} = X$.

(4) Problem 6 on page 99 of Rudin.

Solution. First of all, note that we don't have a specific metric on $E \times f(E)$. Any reasonable metric will do, however, and I can take the sum metric $d((a, b), (c, d)) = d(a, c) + d(b, d)$ for the purpose of this problem. By the same sort of ϵ-δ arguments done for $\mathbb{R}^k$ in Rudin 4.10, a function into $E \times f(E)$ is continuous if and only if each of its component functions are continuous.

First suppose that f is continuous. Then the map $g : E \to E \times f(E)$ given by $x \mapsto (x, f(x))$ is continuous by continuity of f. Because E is compact, $g(E)$ is compact by Rudin 4.14. But $g(E)$ is exactly the graph of f, so we're done.

Now suppose that the graph of f is compact. We define g as before, so that our graph is equal to $g(E)$. Let $h : g(E) \to E$ be the function given by projecting $g(E)$ onto the x-coordinate, that is, $h(x, f(x)) = x$. This function h is clearly continuous because $d(h(a), h(b)) \leq d(a, b)$ for any $a, b \in g(E)$. Also, h and g are inverses by definition. Because we are given that $g(E)$ is compact, we can apply Rudin 4.17 to concluse that the inverse mapping to h, namely g is continuous. But if the function $g(x) = (x, f(x))$ is continuous, its second component, namely $f(x)$, must also be a continuous function, as desired. $\square$

(5) Problem 14 on page 100 of Rudin.

Solution. Pretty much everyone got this problem: apply the Intermediate Value Theorem to $f(x) - x$. We have $f(0) - 0 \geq 0$, $f(1) - 1 \leq 0$, so by IVT, there exists x with $f(x) - x = 0$. Presto: x is a fixed point! $\square$