Math 25b Homework 2

Ivan Corwin and Alison Miller

1 Ivan's problems

(1) Problem 8 on page 99 of Rudin.

Solution. There exists some $\delta > 0$ such that for $x, y \in E$ with $|x - y| < \delta$ we have $|f(x) - f(y)| < 1$.

As E is bounded, we know that $\bar{E}$ is closed and bounded, thus compact. So, $\bar{E} \subset \{N_\delta(x)\}_{x \in E}$ (note E, not $\bar{E}$). Take a finite subcover of this open cover, $N_\delta(x_0), \ldots, N_\delta(x_n)$. Then, for every $x \in E$ we have $x \in N_\delta(x_k)$ for some k. [Note: This gives a proof of the fact that a bounded subset, E, of a compact metric space, (K, d), is in fact *totally bounded* – That is, for any $\delta > 0$, we can take a finite set of points $e_0, \ldots, e_m \in E$ such that for any $e \in E$ we have $d(e, e_k) < \delta$ for some k].

Then, let $M = \max_{i=0}^{n} |f(x_i)|$, and we have that $|f(x)| \leq |f(x_k)| + |f(x_k) - f(x)| < M + 1$ (letting k be such that $|x - x_k| < \delta$). So, f is bounded on E.

To see that boundedness is necessary, consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x$. We note that f is uniformly continuous on $\mathbf{R}$ but not bounded. □

I also have some general comments about this problem. Some people did this proof constructively going straight from the definition of boundedness. The issue there is that it become a little trickier but if done correctly you got high marks. Also, other people used the result that a continuous function maps a compact set to another compact set, which in real space is closed and bounded. People did reasonably well with organizing their ideas here, though sometimes people would have been well served to define variables. Whatever you do, remember, however, not to use madeup ideas. You math should be real and rigorous.

(2) Problem 17 on page 100 of Rudin.

Solution. Let $E = \{x \in (a, b) | f(x-) < f(x+)\}$, $F = \{x \in (a, b) | f(x-) > f(x+)\}$, $G = \{x \in (a, b) | f(x-) = f(x+) < f(x)\}$, $H = \{x \in (a, b) | f(x-) = f(x+) > f(x)\}$.

To each $x \in E$, associate a triple $(p, q, r) \in \mathbf{Q}^3$ satisfying $f(x-) < p < f(x+)$ (which we can choose as $\mathbf{Q}$ is dense in $\mathbf{R}$), $a < q < t < x$ implies $f(t) < p$ (which we can choose by the definition of the left-hand limit), $x < t < r < b$ implies $f(t) > p$ (by definition of right-hand limit). We claim that this mapping is injective. Say $x < x'$ and we associate the same triple (p, q, r) to both. Then, take some $t \in (x, x')$, and we have $a < q < x < t < x'$ so $f(t) > p$ but also $x < t < x' < r < b$ so $f(t) > p$, yielding a contradiction. So, E is at most countable. Similarly, F is at most countable.

To each $x \in G$, associate a triple $(p, q, r) \in \mathbf{Q}^3$ satisfying $f(x-) < p < f(x)$, $a < q < t < x$ implies $f(t) < p$, $x < t < r < b$ implies $f(t) < p$. Then, we claim that this mapping is injective, for if $x < x'$ and we associate the same triple (p, q, r) to both, then we note that $a < q < x < x'$, so $f(x) < p$, but $f(x) > p$, yielding a contradiction. So, G (similarly H) is at most countable.

Then, the set of simple discontinuities, $E \cup F \cup G \cup H$ is a finite union of at most countable sets, and thus at most countable. □

(3) Problem 18 on page 100 of Rudin.

Solution. Fix $\epsilon > 0$. For any $x_0 \in \mathbf{R}$, let $n_0 > \frac{1}{\epsilon}$, and $m_0 = n_0!$. Then, let a be the greatest integer strictly less than $x_0 m_0$, and b the greatest integer strictly greater than $x_0 m_0$ (so $\frac{a}{m_0} < x_0 < \frac{b}{m_0}$, $b - a$ is either 1 or 2, with 2 iff x_0 is rational with denominator dividing m_0). Now, let $\delta < \min(|\frac{b}{m_0} - x_0|, |x_0 - \frac{a}{m_0}|)$. Note that $N_\delta(x_0)$ contains no rational numbers with denominator a factor of $m_0 = n_0!$, and so for $x \in N_\delta(x_0)$ we have $f(x) < \frac{1}{n_0} < \epsilon$. So, $\lim_{t \to x_0} f(t) = 0$.

As $f(x_0) = 0$ for $x_0 \in \mathbf{R} \setminus \mathbf{Q}$, f is continuous at every irrational number. As $f(x_0) = 1$ for $x_0 \in \mathbf{Q}$, f has a simple discontinuity at every rational. □

(4) Let (X, d) be a metric space. A function $f : X \to X$ is called a *contraction* if and only if there exists $\alpha \in [0, 1)$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \quad \text{for all } x, y \in X.$$

A point $p \in X$ is called a *fixed point* if $f(p) = p$.

(a) Prove that every contraction of a complete metric space has a unique fixed point. *(This is known as the Contraction Mapping Theorem.)*

(b) Can you find an example of a function $g : X \to X$ satisfying

$$(\star) \qquad d(g(x), g(y)) < d(x, y) \quad \text{for all } x, y \in X \text{ with } x \neq y$$

with no fixed point? Can such a function have several different fixed points?

(c) If X is compact and $g : X \to X$ satisfies $(\star)$, show that g has a unique fixed point.

(d) Give an example of a compact metric space X and $g : X \to X$ satisfying $(\star)$ such that g is not a contraction.

Solution.

(a)

Lemma. *Let (X, d) be a metric space. Then, for $f : X \to X$ a contraction mapping, f is continuous.*

Proof. Fix a sequence $\{x_i\} \in X$ such that $\lim_{n\to\infty} x_i = x$, and $\epsilon > 0$. Take N such that for $n > N$ implies $d(x_n, x) < \epsilon$, then $d(f(x_n), f(x)) \leq \alpha\epsilon < \epsilon$. So, $\lim_{n\to\infty} f(x_n) = f(x)$, and f is continuous. □

For uniqueness: Say there are distinct $x_0, x_1 \in X$, with $f(x_0) = x_0$ and $f(x_1) = x_1$. Then, $d(x_0, x_1) = d(f(x_0), f(x_1)) < cd(x_0, x_1)$. This yields a contradiction.

For existence: Take any $x_0 \in X$. Let $x_n = f(x_{n-1})$ for $n \geq 1$. Note that $d(x_n, x_{n+1}) = d(f^n(x_0), f^n(x_1)) \leq c^n d(x_0, x_1)$. Then, for $n < m$ note that $d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) = d(x_0, x_1)\left(c^n + c^{n+1} + \cdots + x^{m-1}\right) < c^n \frac{d(x_0,x_1)}{1-c}$. As $\frac{d(x_0,x_1)}{1-c}$ is a finite constant, for any $\epsilon > 0$, there is some $N > 0$ such that $c^n \frac{d(x_0,x_1)}{1-c} < \epsilon$ for $n > N$. So, $\{x_n\}$ is a Cauchy sequence, and has a limit $x \in X$.

Then, $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x)$ (the last equality by the continuity of f).

This was done in section and in the book – oops. The important idea is the iteration, and the fact that for $\alpha < 1$ you can bound the sums you get (yes, you have to look at the sums), combined with continuity of f.

(b) Let $X = [1, \infty)$, and $g(x) = x + \frac{1}{x}$, which we note works as $d(g(x), g(y)) = |x - y + \frac{1}{x} - \frac{1}{y}| = \left|\frac{(x-y)(xy-1)}{xy}\right| = d(x, y)\left(\frac{xy-1}{xy}\right) < d(x, y)$.

Such a function *cannot* have several different fixed points. For say we have $x_0, x_1 \in X$, both fixed points, then $d(x_0, x_1) = d(f(x_0), f(x_1)) < d(x_0, x_1)$, a contradiction.

(c) Uniqueness follows by the observation at the end of (b).

For existence: Define $f(x) = d(g(x), x)$. Note that for any $\epsilon > 0$, $x \in X$, $t \in N_\delta(x)$ for $\delta = \frac{\epsilon}{2}$, we have $|f(x) - f(t)| = |d(g(x), x) - d(g(t), t)| \leq d(x, t) + d(g(t), g(x)) \leq 2d(x, t) < 2\delta < \epsilon$. Now, f is continuous and X compact, so $f(X) \subset \mathbf{R}$ is compact, thus closed and bounded. So f attains its minimum at some $x_0 \in X$. But then, if $x_0 \neq g(x_0)$ we'd have $f(g(x_0)) = d(g(g(x_0)), g(x_0)) < d(g(x_0), x_0) = f(x_0)$, violating the fact that f attains its minimum at x_0. So, we must have $x_0 = g(x_0)$, and so x_0 a fixed point of g.

The iteration argument from (a) doesn't go through unchanged because you can't bound the sums, and just taking a convergent subsequence doesn't immediately work (you can probably fix it up to work, but that requires some extra effort).

(d) Take $X = [0, 1]$ and $g(x) = \frac{1}{2}x^2$. Then, $d(g(x), g(y)) = \frac{1}{2}|x^2 - y^2| = \frac{1}{2}|x - y||x + y| = \frac{x+y}{2}d(x, y) < d(x, y)$, so g satisfies $(\star)$. But, letting $x(c) = c$, $y(c) = 1$, we have that $d(g(x(c)), g(y(c))) = \frac{x(c)+y(c)}{2}d(x(c), y(c)) > cd(x(c), y(c))$ for any $c < 1$, so g is not a contraction on $[0, 1]$.

□