

# Math 25b Homework 3

Due Wednesday 22nd February 2006.

## 1 Alison's problems

(1) Problem 2 on page 114 of Rudin

*Solution.* We first show  $f$  strictly increasing. For any  $x_1, x_2 \in (a, b)$  with  $x_1 < x_2$  we can use the Mean Value Theorem to find  $x \in (x_1, x_2)$  such that

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By assumption  $f'(x) > 0$  and also  $x_2 - x_1 > 0$ , so  $f(x_2) - f(x_1) > 0$ . Hence  $x_1 < x_2$  implies  $f(x_1) < f(x_2)$ , and  $f$  is strictly increasing.

We now note that  $f$  is continuous, so it maps the connected interval  $(a, b)$  to another connected subset of  $\mathbb{R}^2$ , which must also be an interval. Because  $f$  is strictly increasing and  $(a, b)$  has neither a greatest nor a least element,  $f(a, b)$  also has neither a greatest nor a least element, so  $f(a, b)$  is some open interval  $(c, d)$  (where the endpoints may be at  $-\infty$  or  $\infty$ ). Then  $f$  is a bijection between  $(a, b)$  and  $(c, d)$ , so we can define an inverse function  $g : (c, d) \rightarrow (a, b)$ . We now show that  $g$  is continuous, using the open set definition of continuity. We first show that preimages of open intervals are open: for any open interval  $(x_1, x_2) \subset \mathbb{R}$ ,  $g^{-1}(x_1, x_2) = f(x_1, x_2)$  which is an open interval by the same argument as above. Any open set  $U$  in  $(c, d)$  can be written as a union of open intervals, so  $g^{-1}(U)$  is the union of the open preimages of those open intervals, which is also open. This shows that  $g$  is continuous.

We are now ready to show that  $g : (c, d) \rightarrow (a, b)$  is differentiable with derivative  $g'(f(x)) = \frac{1}{f'(x)}$ . Take any  $y \in (c, d)$ . For  $y_1 \neq y$ ,

$$\frac{g(y_1) - g(y)}{y_1 - y} = \frac{g(y_1) - g(y)}{f(g(y_1)) - f(g(y))}.$$

Let  $x = g(y)$ , so  $y = f(x)$ . By the definition of the limit, we know that  $\lim_{x_1 \rightarrow x} \frac{f(x_1) - f(x)}{x_1 - x} = f'(x)$ . By continuity of  $g$ ,  $\lim_{y_1 \rightarrow y} g(y_1) = g(y)$ . Letting  $x_1 = g(y_1)$  and taking a composition of the limits, we see that  $\lim_{y_1 \rightarrow y} \frac{f(g(y_1)) - f(g(y))}{g(y_1) - g(y)} = f'(x)$ . Taking reciprocals and using the above equality, we get

$$g'(f(x)) = g'(y) = \lim_{y_1 \rightarrow y} \frac{g(y_1) - g(y)}{y_1 - y} = 1/f'(x)$$

as desired. □

*Comments:* First of all a nit-pick. In order to know that  $g'$  is even defined at  $y$  the way Rudin does it, you need to check that  $g$  is defined on an interval containing  $y$ . Second, you need to take care when changing variables in your limit. It is possible to have functions  $f_1, f_2$  with  $f_1$  continuous, where  $\lim_{x_1 \rightarrow x} f_2(f_1(x)) = z$  but  $\lim_{y_1 \rightarrow f(x)} f_2(y_1)$  does not exist. The problem is that the first limit only tells you that  $f_2(y_1)$  approaches  $z$  if  $y_1$  approaches  $f(x)$  in the same way that  $f(x_1)$  approaches  $f(x)$ : if  $f$  does not map a neighborhood of  $x$  to a full neighborhood of  $f(x)$ , there may be other ways in which you can have  $y_1$  approach  $y$  that are unrelated to the limit  $\lim_{x_1 \rightarrow x} f_2(f_1(x))$ . Because increasing functions on intervals in  $\mathbb{R}$  are nice, things turn out to be okay in our situation (basically because we have the well-defined continuous inverse  $g$  already) – however, it is important to take care when handling limits.

(2) Problem 3 on page 114 of Rudin.

*Solution.* Choose  $\epsilon < 1/M$  so that  $f'(x)$  is everywhere positive. Everyone did fine on this problem.  $\square$

(3) (a) Problem 9 on page 115 of Rudin. Let  $f$  be a continuous real function in  $\mathbb{R}$ , and  $f'(x)$  exists for all  $x \neq 0$  and  $f'(x) \rightarrow 3$  as  $x \rightarrow 0$ . Does it follow that  $f'(0)$  exists?

*Solution:* YES.

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f'(h)}{1} = 3.$$

Here the first equality is by definition, the second using L'Hopital's rule (as  $f(h) - f(0) \rightarrow 0$  and  $h \rightarrow 0$  as  $h \rightarrow 0$ ) and the third equality by assumption.

*Note:* The chief place you lost points for style was for writing too much. People also lost points if they did not mention L'Hopital or did not justify its use.

(b) Problem 11 on page 115 of Rudin. Suppose  $f$  is defined in a neighborhood of  $x$  and  $f''(x)$  exists.

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} \\ &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x) + f'(x) - f'(x-h)}{2h} \\ &= \frac{f''(x) + f''(x)}{2} = f''(x) \end{aligned}$$

Here the first equality is by L'Hopital's (as numerator and denominator tend to 0 as  $h \rightarrow 0$ ). The third equality uses the definition of derivative (with  $h$  and  $-h$ ).

A good example with limit existing, even if derivative  $f''(x)$  does not, is the function

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0. \end{cases}$$

Here  $\lim_{h \rightarrow 0} \frac{h^2 - h^2 + 0}{h^2} = 0$  but  $f''(0)$  does not exist (+2 from right and -2 from left of 0).

*Note:* Many of you went wrong by not using L'Hopital's rule in this question. The fact that the denominator goes to 0 really matters! Also note that for full points you should have given good reasons for both the limit existing and  $f''(x)$  not existing for the example function. Many of you

*chose examples which weren't even continuous or had a first derivative at the point in question. Take a note of the example above.*

(4) Problem 4 on page 114 of Rudin.

*Solution.* You all know how to do this: apply Mean Value Theorem to  $C_0X + C_1X^2/2 + \dots + C_nX^n/(n+1)$ .  $\square$