Math 25b Homework 3

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1 Ivan's problems

(1) Problem 13 parts (a)-(e) on page 115 of Rudin. (Hint: make sure you make use of what you learn in each part.)

Solution.

(a) Being a composition and product of continuous functions, it suffices to check that f is continuous at 0. If $a = 0$, then $\sin(|x|^{-c})$ does not converge as $x \to 0$. If $a < 0$, then the sin term is bounded, but x^a is unbounded. If $a > 0$, then both terms are bounded, and $x^a \to 0$.

(b) $f'(0) = \lim_{h\to 0} f(h)/h = \lim_{h\to 0} h^(a-1)\sin(|h|^{-c}$. By the reasoning in the first part, this exists iff $(a-1) > 0$, or $a > 1$. The limit, when it exists, is 0 since the sine term is bounded but we have a positive power of h.

(c) We first assume that $a > 1$, since otherwise $f'(0)$ does not exist and thus cannot be bounded. $f'(x)$ in general is $ax^{a-1}\sin(|x|^{-c}) - cx^(a-c-1)\cos(|x|^{-c})$. We consider the behavior as $x \to 0$. Suppose $a < 1+c$, then the first term is bounded ($ax^(a-1)$ is bounded, and the trig term is bounded by 1), but the second term has a negative power of x and therefore becomes unbounded as $x \to 0$. If $a \geq 1+c$, then both terms are bounded, since positive powers of x are bounded and the trig terms are bounded.

(d) Again, we only have to check the continuity at 0. Considering that f' becomes unbounded at 0 if $a < 1+c$, we only have to check cases where $a \geq 1+c$. Suppose $a = 1+c$, then the second term becomes $c\cos(|x|^{-c})$, which does not have a limit. If $a > 1+c$, however, both terms have a bounded trig term and a positive power of x which goes to 0, which by the second part is $f'(0)$, so we have continuity.

(e) $f''(0) = \lim_{h\to 0} f'(h)/h = \lim h \to 0 a x^{a-2}\sin(|x|^{-c}) - cx^(a-c-2)\cos(|x|^{-c})$. The limit of the first exists iff $a > 2$, and the second iff $a > c+2$. Since the two $a-2 > a-c-2$, the two terms cannot both diverge and have a convergent sum (I leave this as an exercise to the reader). Since the second condition is stronger than the first, we have convergence iff $a > c+2$.

□

Note: Some people were too technical in this and went to ϵ and δ proofs. Continuity at a point just means that the left and right limits agree. Also, people rigorously proved that $\sin(x^{-c})$ for $c > 0$ oscillated as x goes to 0. We know this, we did it in class and so you don't need to show it every time. If you want to prove this, then at least write it as a Lemma, prove it once, and then appeal to the Lemma in the future. Also, almost everyone did not deal with the issue of terms involving cos and sin canceling. Of course this doesn't happen because of the different phases of the two functions, however, without this it is possible that they could cancel, leading to complete cancelation even when each term diverges on its own. I took of 2 points on every pset for this.

(2) Problem 25 on page 118 of Rudin.

Solution. (a) This is where the tangent line at $x, f(x)$ to f strikes the x-axis.

(b) We strengthen the inductive argument, namely that $x_n > \xi$ and is decreasing. $x_{n+1} - x_n = -f(x_n)/f'(x_n)$ then gives us the extra information that $f(x_n) > 0$, so the difference is negative. i.e. $x_{n+1} \leq x_n$. Also, note that for some $a \in (\xi, x_n)$, $f'(a) = (f(x_n) - f(\xi))/(x_n - \xi)$, and $a < x_n \to f'(a) < f'(x_n) \to f(x_n)/f'(x_n) < x_n - \xi$.

Since x_n are bounded and monotonic, it converges to some L. $\lim x_{n+1} = \lim x_n$ gives $L - f(L)/f'(L) = L$, so $f(L) = 0$. Therefore, the only possible limit is ξ.

(c) Taylor gives the existence of a t_n in the given interval s.t. $0 = f(x_n) + (\xi - x_n)f'(x_n) + f''(t_n)/2(\xi - x_n)^2$. After algebra this becomes $1/2f''(t_n)(x_n - \xi)^2/f'(x_n) = x_{n+1} - \xi$, as desired.

(d) Induct on $x_{k+1} \leq 1/A[A(x_1 - \xi)]^{2^k}$. For $k = n$ this follows immediately from the previous part, bounding by M and δ where the derivatives appear. I spare you the algebra here.

(e) g is fixed when $f(x)/f'(x) = 0$, which is what we want. As $x \to \xi$, just use the quotient rule to show that $g' \to 0$.

(f) It diverges. This is because the second derivative is not bounded, not because the method is false.

□

Note: Lots of people didn't get what geometric interpretation meant. It didn't mean to draw a picture necessarily. It meant to talk in terms of lines and coordinate systems. This question is a little stupid since there is no rigorously developed meaning to giving a geometric interpretation, however I think it was reasonable to ask it. Plus, an interpretation should mean in words, not just numbers and symbols. Also some people didn't address why the last example didn't violate Newton.

(3) Construct a function $f : \mathbf{R} \to \mathbf{R}$ such that

- f is infinitely differentiable (i.e. the nth dertivative $f^{(n)}(x)$ exists for all $n \in \mathbf{N}$ and all $x \in \mathbf{R}$);
- $f(x) = 0$ for all $x \leq 0$;
- f is not the zero function.

Can you find such a function satisfying $f(x) = 1$ for $x \geq 1$?

Solution. Consider $f(x) = e^{-1/x}$ for $x > 0$, and 0 otherwise. To show it is infinitely differentiable, it suffices to show so at 0, i.e. $\lim_{x \to 0} f^{(n)}(x)/x = 0$ for all n. It is easy to show inductively that this is the quotient of a polynomial $P(1/x)$ of degree at most $n+1$ and $e^{1/x}$. Rudin in chapter 3 shows that this goes to 0 as $x \to 0$.

Now, to construct the second function, we can reuse this function. Consider $g(x) = f(x) + f(1-x)$. For $x < 0$, this is going to have the value $f(1-x)$. For $x > 1$, this is going to take the value $f(x)$. Now, consider $h(x) = f(x)/g(x)$. It will then take the value 0 for $x < 0$ but $f(x)/f(1-x) = 1$ for $x > 1$. A slight revision of Rudin 5.3 shows that these functions are all infinitely differentiable, and we are done. □

Note: People neglected showing higher derivatives match sometimes. Use L'Hopital to do this. Then some people appealed to properties of graphs which they drew. BIG DEAL: WE CAN NEVER APPEAL TO WHAT A GRAPH LOOKS LIKE, BUT RATHER MUST USE RIGOROUS IDEAS LIKE BOUNDEDNESS OR COMPACTNESS OR CONTINUITY.

2 The problems I didn't assign—good practice!

(1) Problem 1 on page 114 of Rudin.
(2) Problem 10 on page 115 of Rudin.
(3) Problem 14 on page 115 of Rudin. (Another convexity problem.)
(4) Problem 22 on page 117 of Rudin. (More fixed point problems.)
(5) Problems 26—29 on page 119 of Rudin. (Problems on ODE's—I might set these later in the course.)