

Math 25b Homework 6 Solutions Part 1

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As usual, solutions only to selected problems.

1 Alison's problems

(4) Problem 13 on page 167 of Rudin.

Solution. (a) We follow the outline provided in Rudin's hint:

(i) The sequence f_n is pointwise bounded by 0 and 1 at any point, and the rationals are countable, so Rudin 7.23 says that we can pick out a subsequence $\{f_{n_i}\}$ such that the sequence $\{f_{n_i}(r)\}$ is convergent for any rational r . Denote this limit by $f(r)$: note that $f(r)$ is a limit of monotonically increasing functions on \mathbb{Q} , so it too is monotonically increasing.

(ii) We extend f to all of \mathbb{R} as follows: for general $x \in \mathbb{R}$, we define $f(x) = \sup\{f(r) \mid r \in \mathbb{Q}, r \leq x\}$. This definition is consistent with our previous definition of f on \mathbb{Q} because that f was monotonically increasing on \mathbb{Q} . Our new f is also a monotonically increasing function of x by definition.

(iii) This is the tricky part: as Rudin suggests, it's really important to use monotonicity properly here – some people tried to do it without, and the ϵ -arguments don't go through in that case.

Suppose that f is continuous at x . We need to show that $f_{n_i}(x) \rightarrow f(x)$: that is, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that if $i > N$, $|f_{n_i}(x) - f(x)| < \epsilon$. First, use continuity of f at x to choose δ such that if $|y - x| < \delta$, $|f(y) - f(x)| < \epsilon/2$. Then use the fact that \mathbb{Q} is dense in \mathbb{R} to pick rationals r, s with $r \in (x - \delta, x)$, $s \in (x, x + \delta)$. We know from (i) that $f_{n_i}(r) \rightarrow f(r)$, $f_{n_i}(s) \rightarrow f(s)$, so choose N large enough that if $i > N$, both $|f_{n_i}(r) - f(r)| < \epsilon/2$ and $|f_{n_i}(s) - f(s)| < \epsilon/2$. Using the triangle inequality, we then find that

$$|f_{n_i}(r) - f(x)| < |f_{n_i}(r) - f(r)| + |f(r) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Similarly, $|f_{n_i}(s) - f(x)| < \epsilon$, so. Now we use this along with the fact that f_{n_i} is increasing to give us the following string of inequalities:

$$f(x) - \epsilon \leq f_{n_i}(r) \leq f_{n_i}(x) \leq f_{n_i}(s) \leq f(x) + \epsilon.$$

So $f_{n_i}(x) \in B_\epsilon(f(x))$, as desired. Since this works for all $n_i > N$, and ϵ was arbitrary, $\{f_{n_i}(x)\} \rightarrow f(x)$.

(d) Because f is monotonic increasing, f has countably many points of discontinuity, so, by 7.23 again, we can find a subsequence of $\{f_{n_i}\}$ that converges at every such point (it may not converge to our original function $f(x)$). Since $\{f_{n_i}(x)\}$ converges to $f(x)$ at points of continuity of f , the subsequence also converges at any point of continuity of f , so it converges pointwise everywhere.

(b) The argument of (iii) also shows that, with x , r , s defined as above, for any $y \in (r, s)$, $f_{n_i}(y) \in B_\epsilon(f(x))$ (replace x with y in the last string of inequalities). Also, $|x - y| < \delta$, so $|f(x) - f(y)| < \epsilon/2$ and by the triangle inequality, $|f_{n_i}(y) - y| < 3\epsilon/2$. Since the n_i was independent of our choice of $y \in (r, s)$, this means that $\{f_{n_i}\} \rightarrow f$ uniformly on (r, s) . Because f is now everywhere continuous, we can do this for any x in our compact set C , and find a neighborhood of x on which $\{f_{n_i}\} \rightarrow f$ uniformly. Cover C by such neighborhoods: by compactness, we can take a finite subcover. Since $\{f_{n_i}\}$ converges uniformly on each of the finitely many neighborhoods, $\{f_{n_i}\}$ also converges uniformly on their union, which covers C , so $\{f_{n_i}\}$ converges uniformly to f on C . \square