

Math 25b Homework 7 Solutions Part 1

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As usual, solutions only to selected problems.

1 Alison's problems

(4) *Differentiating bilinear functions* Problem 2-12 on page 23 of Spivak.

Solution. (a) This part is a bit tricky. In some sense, it boils down to showing that if $|(h, k)|$ is bounded, then $|f(h, k)|$ is bounded, and then noting that if we multiply (h, k) by ϵ , $|f(h, k)|$ is multiplied by ϵ^2 but $|(h, k)|$ is only multiplied by a factor of ϵ . However, proving the first part is a bit tricky. We already know that linear functions are continuous, but we don't yet know that bilinear functions are continuous, so we can't use that: continuity of bilinear function tells us that for fixed k , if h is bounded, so is $|f(h, k)|$, but that isn't enough. What we'll do instead is just estimate everything in terms of a basis, the same way that we showed that linear functions are continuous.

Write h, k in terms of the standard bases for \mathbb{R}^n and \mathbb{R}^m : $h = \sum_{i=1}^n h^i e_i$, $k = \sum_{j=1}^m k^j e_j$. We can use bilinearity to expand $f(h, k)$:

$$f(h, k) = \sum_{i=1}^n \sum_{j=1}^m h^i k^j f(e_i, e_j).$$

Let $M = \max_{i,j} |f(e_i, e_j)|$. By the triangle inequality,

$$|f(h, k)| \leq \sum_{i=1}^n \sum_{j=1}^m |h^i k^j f(e_i, e_j)| \leq M \sum_{i=1}^n \sum_{j=1}^m |h_i| |k_j|.$$

Note that the h_i 's and the k_j 's are components of the vector (h, k) , so for each i , $|h_i| \leq |(h, k)|$ and for each j , $|k_j| \leq |(h, k)|$. (This is a really crude bound, but it works, so it's okay.) Plugging this into the previous equation gives

$$|f(h, k)| \leq Mmn|(h, k)|^2.$$

Then

$$\frac{|f(h, k)|}{|(h, k)|} \leq \frac{Mmn|(h, k)|^2}{|(h, k)|} = Mmn|(h, k)|.$$

As $(h, k) \rightarrow 0$, $Mmn|(h, k)| \rightarrow 0$, so $\frac{|f(h, k)|}{|(h, k)|} \rightarrow 0$ as well.

b) After this, everything is a breeze. Use the definition of derivative, and use bilinearity to write $f(a + h, b + k) - f(a, b) - f(a, k) - f(h, b) = f(h, k)$. Divide by $|(h, k)|$ and take limits as $(h, k) \rightarrow 0$.

c) Multiplication is bilinear, and part b) looks a lot like 2-3 part (5).

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