

# Math 25b Homework 8

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## 1 Ivan's Problems

Special thanks to Yifei Chen for providing the tex for these solutions.

- (a) If  $f : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$  is smooth and  $\frac{\partial f}{\partial z}(a, b, c) \neq 0$ , then we may apply the Implicit Function Theorem to  $f(a, b, c) - k$  to see that there is an open set  $U \subset \mathbb{R}^2$  containing  $(a, b)$  and a differentiable function  $g : U \rightarrow \mathbb{R}$  such that  $f(a, b, g(a, b)) - k = 0$ . The points in the set  $\{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = k\}$  sufficiently close to  $(a, b, c)$  form a smooth surface in  $\mathbb{R}^3$  because since  $f$  is differentiable at  $(a, b, c)$ ,  $f(x, y, z) = k$  can be approximated locally there by a tangent plane.
- (b) Again, near every point  $(a, b, c) \in f^{-1}(k)$ , the set  $f^{-1}(k)$  looks like a smooth surface in  $\mathbb{R}^3$  because the Implicit Function Theorem guarantees that  $f(x, y, z) = k$  can be written as  $z = g(x, y)$ , and since all of the partial derivatives exist at  $(a, b, c)$ ,  $f(x, y, z)$  can be approximated by the tangent plane  $z - c = g_x(a, b)(x - a) + g_y(a, b)(y - b)$ .
- (c) Consider any smooth curve  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$  defined by

$$\gamma(t) = (x(t), y(t), z(t))$$

such that  $\gamma(t) \in f^{-1}(k)$  for all  $t$  and  $\gamma(0) = (a, b, c)$ . Since  $\gamma \in f^{-1}(k)$ , we have  $f(x(t), y(t), z(t)) \equiv k$ . Differentiate both sides with respect to  $t$  and apply the Chain Rule to yield  $\langle \nabla f, \gamma'(t) \rangle = 0$ . Hence,  $\langle \nabla f(a, b, c), \gamma'(0) \rangle = 0$  so that  $\nabla f(a, b, c)$  is perpendicular to the surface  $f^{-1}(k)$  at the point  $(a, b, c)$ .

- (d) See the attachment. I'm a bad artist, but the level curves are spheres centered at  $(0, 0, 0)$ . We have  $\nabla f(x, y, z) = (2x, 2y, 2z)$ .

2. (a) Suppose that  $f'(a) \neq 0$  for all  $a \in \mathbb{R}$  and that  $f$  is not injective. Then there exist  $a, b \in \mathbb{R}$  such that  $a \neq b$  and  $f(a) = f(b)$ . But by Rolle's Theorem, there exists  $c \in (a, b)$  such that  $f'(c) = 0$ . Contradiction. Hence  $f$  is injective.
- (b) We have that

$$f'(x, y) = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}.$$

Hence  $\det f'(x, y) = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \neq 0$  for all  $(x, y) \in \mathbb{R}^2$ . However,  $f$  is not injective since  $f(0, 0) = f(0, 2\pi) = (1, 0)$ . Now, if  $\cos x = \cos y$  and  $\sin x = \sin y$ , then  $y = x + 2\pi k$ . Hence, noting that  $e^x$  is injective, for every  $(a, b) \in \mathbb{R}^2$ , we have that  $f|_{B_\pi(a,b)}$  is injective.