Math 25b Homework 9b solutions

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1 Ivan's problems

(1) *Lagrange Multipliers*

Let f and g be smooth functions from $\mathbb{R}^3 \to \mathbb{R}$. In this question we will develop a method for solving the following *constrained maximization problem*: find the maximum value $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$. (That is only worry about the value of f for points (x, y, z) such that $g(x, y, z) = k$.) We also assume that:

• ∇g is not zero anywhere on $g^{-1}(k)$. (So you know from HW8 that the level set $g^{-1}(k)$ is a smooth surface);

• all level sets of $f^{-1}(l)$ are smooth surfaces[1].

Let M be the maximum value of f on the surface $g^{-1}(k)$ and suppose this maximum occurs at $(a, b, c) \in g^{-1}(k)$. We want to find M.

(a) Show that the level surfaces $f^{-1}(M)$ and $g^{-1}(k)$ are tangent at (a, b, c) **and** that (a, b, c) is a solution to the following system of equations

$$(*) \qquad \begin{cases} \nabla f = \lambda \nabla g(x, y, z) \\ g(x, y, z) = k \end{cases}$$

(**Warning:** please be careful about what you mean by perpendicular, parallel and tangent. Take a look at HW 8 again for the definition of what it means for a vector to be perpendicular to a surface.)

(b) Must any solution to (*) give either the maximum or minimum values of $f(x, y, z)$ on the set $g^{-1}(k)$?

(c) Use the technique of this question to find the minimum surface area of a rectangular box which has volume equal to 8 units.

[1] Note we can relax this assumption a little to "all level sets of f which meet a neighborhood of the surface $g^{-1}(k)$ are smooth surfaces in a neighborhood of $g^{-1}(k)$".

Solution. (a) The gradient of g is perpindicular to the level surface $g^{-1}(k)$ at $p = (a, b, c)$. Let γ be a smooth curve in $g^{-1}(k)$ such that $\gamma(0) = p$. Note that $(f(\gamma(t))'(0) = 0$ since f has a maximum there in $g^{-1}(k)$. However, this is equal to the gradient of f times $\gamma'(0)$, so the gradiant of f is also perpindicular to the same level surface. By the implicit function theorem, write points near p as $(x, y, h(x, y)$ - so in γ we have $(\gamma_x(t), \gamma_y(t), h(\gamma_x(t), \gamma_y(t)))$. The derivatives of this at 0 forms our tangent space, which is a plane. Now, we know that the two gradients are perpindicular to the same plane - so they are actually parallel. So they are both perpindicular to $f^{-1}(M)$. By the same logic, both gradients are perpindicular to the same tangent plane of $f^{-1}(M)$ as well. So we know the two level surfaces are tangent - since they have the same tangent plane.

(b) No. Try something like $f(x, y, z) = z$ and $g(x, y, z) = x^3 - z$. Then $p = (0, 0, 0)$ is in $g^{-1}(0)$. The gradient of f gives $(0, 0, 1)$, and the gradient of g is $(0, 0, -1)$ which is parallel. But $f(0, 0, 0) = 0$, $f(1, 0, 1) = 1$, $f(-1, 0, -1) = -1$, and all evaluate to 0 at g.

(c) Straight lagrange multipliers show you must have

$$2y + 2z = ayz \tag{1}$$
$$2x + 2z = axz \tag{2}$$
$$2z + 2x = axy. \tag{3}$$

One can quickly see thath they must all be 2, which is a minimum. A maximum doesn't exist since you can make the box really really long, such as the triple $(x, 8/x^2, x)$.

□

(2) *Content zero subsets do not affect integrals*

Suppose that $A \subset \mathbb{R}^n$, $B \subset A$ such that $A \setminus B$ has content zero. Suppose that $f : A \to \mathbb{R}$ is a bounded function and that $g : B \to \mathbb{R}$ is the restriction of f to B. Show (i) that $\int_A f$ exists if and only if $\int_B g$ exists and (ii) that if the integrals exist then they are equal.

Solution. Make a rectangle U that contanis A, and thus B. $\int_A f = \int_F f\chi_A$ if the integral exists.

For the first direction, say $\int_A f$ exists. We can refine a partion P (with $U - L < \epsilon$, you know the drill) such that the $A - B$ part is bounded by a *finite* number of rectangles with total volume ϵ. So in the refinement,

$$|U(f\chi_A) - U(f\chi_B)| = |\sum_s |sup_{x\in s}(f(x))\chi_A(x) - sup_{x\in s}(f(x))\chi_B(x)|v(s)|| \tag{4}$$

only gets the finite rectangles we chose, so it is $< M\epsilon$.

We can get the same thing for lower sums, and we know that $|U - L|$ for A is bounded by ϵ. So we get $|U - L|$ for B (with the triangle inequality) is bounded by $M\epsilon + \epsilon + M\epsilon$. So B is integrable.

I leave the other direction as an exercise - it is the same as this one.

If both integrals exist, this process shows us already that they have the same integral, since they only differ by something that can be bounded by $M\epsilon$.

□

(3) *Some integrals*

(a) Prove that if $f(x, y)$ is a continuous function and R is the region

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

where $g_1(x) \leq g_2(x)$ for all $x \in [a, b]$ and the functions g_1 and g_2 are continuous, then

$$\int_R f(x, y)\, dxdy = \int_{x=a}^{x=b} \left(\int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\, dy \right) dx.$$

Such regions are "swept out by vertical lines". Can you state a similar result for regions "swept out by horizontal lines"?

(b) Sketch the region $S = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$ and evaluate

$$\int_S e^{x+y}\, dxdy.$$

(c) Sketch the solid bounded by the surface $z = x^2 - y^2$, the xy-plane and the planes $x = 1$ and $x = 3$ and compute its volume.

Solution. (a) So clearly we are supposed to use Fubini somehow. Choose a rectangle $A \times B$ big enough, with A (the x-axis) of it being $[a, b]$.

The integral we want is then $\int_{A\times B} f(x,y)\chi_R(x,y)dxdy$. Note that $f_x(y) = f(x,y)\chi_R$ is discontinuous at at most two points, those being $g_1(x)$ and $g_2(x)$. We know this is integrable. So by Fubini we have $\int_{A\times B} f\chi_R$ can be written as a double integral, in the exact format we desired.

(b) This thing looks like a diamond. You can now construct a rectangle to include it, such as $[-1,1]^2$. Although you can also do, say, 4 smaller rectangles, though most people did it this way. Anyway, on the big rectangle, you know the f_x, defined same way as the previous problem, is integrable since it is discontinuous at at most two points. A straightforward calculation gives $e - e^{-1}$, which is about 2.35.

(c) It looks like a shell, sort of. By smilar methods, you should get

$$\int_1^3 (\int_{-x}^{x} x^2 - y^2 dy)dx = 80/3. \tag{5}$$

□

(4) *The Five Lemma*

Suppose that each row of the diagram (see pset- I'm having trouble loading this include) is an exact sequence of linear maps between vector spaces, and that $T_1, \ldots, T_5$ are linear maps such that T_1, T_2, T_4, and T_5 are isomorphisms. Show that T_3 is also an isomorphism. (We'll use this lemma later on in HW exercises.)

Solution. I box "defining" properties.

We prove **injectivity**: Say $\boxed{x \in \ker T_3}$. Then, $\varphi_3 T_3 x = 0 = T_4\phi_3 x$, so $\phi_3 x \in \ker T_4$. As T_4 is injective, this means $x \in \ker\phi_3$. As the top row is exact, this means $x \in \operatorname{im}\phi_2$. So, let $y \in V_2$ be such that $\boxed{\phi_2 y = x}$. Then, $0 = T_3 x = T_3\phi_2 y = \varphi_2 T_2 y$, so $T_2 y \in \ker\varphi_2$. By exactness of the bottom row, $T_2 y \in \operatorname{im}\varphi_1$. Let $z \in W_1$ be such that $\boxed{\varphi_1 z = T_2 y}$. As T_1 is surjective, we may take $u \in V_1$ such that $\boxed{T_1 u = z}$. Then, $T_2 y = \varphi_1 T_1 u$, which by the commutativty of the diagram also equals $T_2\phi_1 u$. By the injectivity of T_2, we have $y = \phi_1 u$. Then, $x = \phi_2 y = \phi_2\phi_1 u$, which by exactness of the sequence is 0. This proves injectivity.

[Note that we have used the injectivity of T_2 and T_4 and the surjectivity of T_1.]

Now, we prove **surjectivity**: Say $\boxed{x \in W_3}$. Then, by the surjectivity of T_4 there is a $y \in V_4$ such that $\boxed{\varphi_3 x = T_4 y}$. By the exactness of the bottom row, $0 = \varphi_4 \varphi_3 x = \varphi_4 T_4 y$, and by commutativity this also equals $T_5 \phi_4 y$. Then, by the injectivity of T_5, this means that $y \in \ker \phi_4$. By exactness of the rop row, there must exist a $z \in V_3$ such that $\boxed{\phi_3 z = y}$. Now, by commutativity of the diagram $\varphi_3 x = T_4 y = T_4 \phi_3 z$ must equal $\varphi_3 T_3 z$. So, $T_3 z - x$ must be in $\ker \varphi_3$ which by exactness is $\operatorname{im} \varphi_2$. So, we can take a $u \in W_2$ such that $\boxed{\varphi_2 u = T_3 z - x}$. By surjectivity of T_2, we can take a $w \in V_2$ such that $\boxed{T_2 w = u}$. So, by commutativity, $T_3 z - x = \varphi_2 T_2 w = T_3 \phi_2 w$. So, $T_3 (z - \phi_2 w) = T_3 - (T_3 z - x) = x$. This proves surjectivity.

[Note that we have used the surjectivity of T_2 and T_4 and the injectivity of T_5.] $\square$

2 Supplementary problems — optional.

(1) How and under what conditions can we generalize the method of Lagrange multipliers to solve the following problem: find the extreme values of $f(x, y, z)$ subject to the constraints $g(x, y, z) = k$ and $h(x, y, z) = l$?

(2) There is more on Lagrange multipliers in Spivak, namely Problems 5-16 and 5-17 on page 122.

(3) Need more practice with integrals or any of the multivariable material? Any multivariable calculus textbook will have plenty of examples. One that is in use (and on reserve in Cabot library) is Stewart's Multivariable Calculus textbook.