Math 25a Homework 8 Solutions

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1 Alison's problems

(1) (a) If L, M, and N are subspaces of a vector space V, show that $L \cap (M+N) = (L\cap M)+(L\cap N)$ does not necessarily hold.
(b) Prove that $L \cap (M + (L \cap N)) = (L \cap M) + (L \cap N)$.

Solution. (a)Take $V = \mathbb{R}^2$, and let L, M, N be the lines through the origin: $L = \{(x, x) \mid x \in \mathbb{R}\}$, $M = \{(x, 0) \mid x \in \mathbb{R}\}$, and $N = \{(0, x) \mid x \in \mathbb{R}\}$. Then $M + N$ is all of V, so $L \cap (M + N) = L$. However, $L \cap M$ and $L \cap N$ both contain only 0, so $(L \cap M) + (L \cap N) = \{0\} \neq V = L \cap (M + N)$.

(b) First we show that $L\cap(M+(L\cap N)) \subset (L\cap M)+(L\cap N)$. Suppose that $v \in L\cap(M+(L\cap N))$. Then $v \in L$ and $v \in M + (L \cap N)$, so v can be written as $v_1 + v_2$, $v_1 \in M$, $v_2 \in L \cap N$. Then $v \in L$, $v_2 \in L$, so because L is closed under addition and scalar multiplication, $v+(-v_2) = v_1+v_2-v_2 = v_1$ is also an element of L. So $v_1 \in L \cap M$, $v_2 \in L \cap N$, hence $v = v_1 + v_2 \in (L \cap M) + (L \cap N)$.

For the other inclusion, $(L\cap M)+(L\cap N) \subset L\cap(M+(L\cap N))$, suppose $v \in (L\cap M)+(L\cap N)$. Then we can write $v = v_1 + v_2$, $v_1 \in L \cap M$, $v_2 \in L \cap N$. Because L is a vector space and $v_1, v_2 \in L$, $v = v_1 + v_2 \in L$ also. Additionally, $v_1 \in M$, $v_2 \in L \cap N$, so $vinM + L \cap N$. As a result, $v \in L \cap (M + L \cap N)$, as desired. □

(2) Let S be a subset of a vector space V. Show that the following are equivalent:
(a) S is a basis for V.
(b) S is a maximal linearly independent subset of V.
(c) S is a minimal spanning set in V.
Note: For the purposes of this problem, "maximal linearly independent" means that S is linearly independent and there is no $v \in V \setminus S$ such that $S \cup \{v\}$ is linearly independent. "Minimal spanning set" means that S spans and for no $v \in S$ does $S \setminus \{v\}$ span.

Solution. We first show that (a) is equivalent to (b), and then that (a) is equivalent to (c).

(a) ⇒ (b): Suppose S is a basis for V. Then certainly S is a linearly independent set. To check maximality, suppose that $v \in V$, $v \notin S$. Then $v \in \operatorname{Span} S$, so we can write v as a finite linear combination $v = c_1v_1 + c_2v_2 + \cdots + c_nv_n$ of elements $v_1, v_2, \ldots, v_n$ of S, so the linear combination of elements of $S \cup \{v\}$ given by $-v + c_1v_1 + c_2v_2 + \cdots + c_nv_n$ is zero. Hence $S \cup \{v\}$ is not linearly independent, and S is a maximal linearly independent set.

(b) ⇒ (a): Suppose S is a maximal linearly independent set. Then we only need to show that S also spans. Suppose not: let v be an element of V that is not in $\operatorname{Span} S$. We claim that $S \cup \{v\}$ is linearly independent. For if not, there must be a linear dependence relation of the form $c_1v_1 + c_2v_2 + \cdots c_nv_n = 0$, where $v_1, v_2, \ldots, v_n$ are distinct elements of $S \cup \{v\}$ and the c_i are elements of F, not all 0. Because S by itself is linearly independent, one of the terms must contain

v: without loss of generality, $v = v_1$ and c_1 must be nonzero. Then $v_2, \ldots, v_n \in S$, and we can rewrite our relation as $v = v_1 = (-c_2/c_1)v_2 + (-c_3/c_1)v_3 + \cdots + (-c_n/c_1)v_n$. So $v \in \operatorname{Span} S$; since this is true for any $v \in V$, S spans V. (We've effectively shown a version of the Linear Dependence Lemma that works for infinite sets as well.)

(a) $\Rightarrow$ (c): Suppose that S is a basis for V: S spans by definition. We need to show that S is a minimal spanning set. By way of contradiction, suppose that $S \setminus \{v\}$ spans for some $v \in S$. The original set S is linearly independent, so $S \setminus \{v\}$ is surely linearly independent as well. Hence $S \setminus \{v\}$ is also a basis: by (a) $\Rightarrow$ (b), $S \setminus \{v\}$ is a maximal linearly independent set. But if we add v, we get S, which is also linearly independent, contradicting the maximality of $S \setminus \{v\}$. So instead S must be a maximal spanning set.

(c) $\Rightarrow$ (a): Suppose that S is a minimal spanning set, so S spans, and we need to show that S is linearly independent. Suppose not: then there is some linear dependence relation between elements of S. By the infinite version of the Linear Dependence Lemma argument used in (b) $\Rightarrow$ (a), we can write some $v \in S$ as a linear combination of other elements of S. This means that we can get rid of v, and $\operatorname{Span}(S - \{v\}) = \operatorname{Span}(S)$, contradicting the minimality of S.

□

(3) Problem 10 on page 35 of Axler.

Solution. Let $(v_1, \ldots, v_n)$ be a basis of V. For each j, let U_j be $\operatorname{Span}(v_j)$, which is the same as $\{cv_j \mid c \in F\}$. By definition of $(v_1, \ldots, v_n)$ as a basis of V, each vector $v \in V$ can be written uniquely as $v = c_1v_1 + c_2v_2 + \cdots c_nv_n$. This is exactly what we need to say that $V = U_1 \oplus U_2 \cdots \oplus U_n$. □

(4) Problem 14 on page 36 of Axler.

Solution. We use Axler 2.18 to deduce that $9 \geq \dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 10 - \dim(U \cap W)$. So $\dim(U \cap W) \geq 1$, and $U \cap W$ cannot just be 0. □

(5) (a) Find a statement of Zorn's Lemma.

(b) Use Zorn's Lemma to show that every vector space contains a maximal linearly independent subset S. Note that from 2(b) above, this tells you that S is a basis for V.

Solution. (a) See, for example, Wikipedia (`http://en.wikipedia.org/wiki/Zorns _lemma`):

> Every partially ordered set in which every chain (i.e. totally ordered subset) has an upper bound contains at least one maximal element.

(See the article for definitions of the terms. Also, we technically need this set to be nonempty.)

(b) Let $\mathcal{P}$ be the partially ordered set whose elements are linearly independent subsets of V, where the partial order is by inclusion: $S \leq T \Leftrightarrow S \subset T$. This is a partial order by basic properties of $\subset$, and is certainly nonempty (if nothing else, it contains the empty set). We need to show that it satisfies the upper bound property. Let $\mathcal{C}$ be a totally ordered subset (chain) of $\mathcal{P}$. Then we can construct an upper bound for all the elements of $\mathcal{C}$ by taking the union of them all: $T = \cup_{S \in \mathcal{C}} S$. If this is an element of $\mathcal{P}$, i.e. if it is a linearly independent set, it will be an upper bound for $\mathcal{C}$ in $\mathcal{P}$ by construction.

So we need to check that the elements of T are linearly independent. Suppose not: then we'd have some linear relation of the form $c_1t_1 + c_2t_2 + \cdots c_nt_n = 0$ where $t_1, t_2, \ldots t_n \in T$ and $c_1, c_2, \ldots c_n \in F$, not all 0. For each i, $t_i \in T = \cup_{S \in \mathcal{C}} S$, so $t_i \in S_i$ for some $S_i \in \mathcal{C}$. We then consider the collection $S_1, S_2, \ldots, S_n$ of elements of $\mathcal{C}$: this is totally ordered – without loss of generality it is ordered as $S_1 \subset S_2 \subset \ldots S_n$. Then for each i, $t_i \in S_i \subset S_n$. But then our linear dependence relation $c_1t_1 + c_2t_2 + \cdots c_nt_n = 0$ is a linear dependence of elements of S_n. This is a contradiction, because S_n, being an element of $\mathcal{P}$, is a linearly independent set. So it must instead be the case that T itself is linearly independent.

Now we're almost done. We've checked that $\mathcal{P}$ satisfies the conditions of Zorn's lemma, so the conclusion of Zorn's lemma holds as well for $\mathcal{P}$. That is, $\mathcal{P}$ has a maximal element, that is to say, a maximal independent subset of V. But we showed in problem 2 that a maximal independent subset is a basis, so V has a basis and we're done.

□