Math 272a, homework 10

November 18, 2003

Problem 1. Let M be a compact oriented manifold without boundary, and write $F^p(M)$ for $H^p(M)/\text{torsion}$ — a finitely generated free $\mathbf{Z}$-module. In class, we used Poincaré duality and the universal coefficient theorem to prove that

$$\begin{aligned} Q : F^*(M) \times F^*(M) &\to \mathbf{Z} \\ (a, b) &\mapsto \langle a \smile b, [M] \rangle \end{aligned}$$

is a unimodular bilinear form (a perfect pairing) on $F^*(M) = \oplus F^p(M)$. Here we have written $[M]$ for the fundamental class of M.

Suppose now that M arises (with its given fundamental class) as the boundary of a compact, oriented manifold-with-boundary, W. (We say 'M is an oriented boundary.') Show that Q vanishes on the image of the map $H^*(W) \to H^*(M)$.

Show that the rank of this image is half the rank of $H^*(M)$. (Use the universal coefficient theorem, and exercise 34 from section 3.3 of Hatcher's book.)

Problem 2. In the notation of the first question, suppose now that the compact, oriented manifold M has dimension $4k$. The form Q, defined in the first question, gives a map

$$Q : F^{2k}(M) \times F^{2k}(M) \to \mathbf{Z}$$

which is a *symmetric* bilinear form, because $a \smile b = b \smile a$ when at least one of a and b has even dimension. The *signature* of the oriented manifold M is defined to be the signature of this symmetric bilinear form.

What is the signature of $\mathbf{CP}^2$ with its complex orientation? What is the signature of $S^2 \times S^2$?

Suppose that M is an oriented boundary, as in the first question. Show that the signature of M is zero.

Deduce that the disjoint union of any number of copies of $\mathbf{CP}^2$, each with the same orientation, is never an oriented boundary. (Note however that $[0, 1] \times \mathbf{CP}^2$ is an orientable 5-manifold having two copies of $\mathbf{CP}^2$ as its boundary: the two copies come with *opposite* orientations.)

Problem 3. Show that $H_c^{p+1}(X \times \mathbf{R})$ is isomorphic to $H_c^p(X)$, for any topological space X.

Problem 4. Let E be an infinite-dimensional normed vector space (e.g. $C^0[0, 1]$). Show that $H_c^*(E)$ is zero.

Problem 5. Let E be as above, and let $P \subset E$ be a 2-dimensional linear subspace (over $\mathbf{R}$). Let $K \subset P$ be a standard circle. What are the ordinary homology groups of $E \setminus K$?

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