Math 272a, homework 11

December 7, 2003

Problem 1. We use X^Y to denote the set of continuous maps from Y to X. Read about the compact-open topology for X^Y in the appendices to Hatcher's book, and also about the conditions under which $X^{Y\times Z}$ is homeomorphic to $(X^Y)^Z$.

Let (X, x_0) be a path-connected, pointed space, and let PX be the *path-space* of X, the space

$$PX = \{\, \gamma \in X^I \mid \gamma(1) = x_0 \,\}.$$

Prove that PX is contractible. Prove that the map $p : PX \to X$ given by $p(\gamma) = \gamma(0)$ is a Serre fibration.

Now suppose X is a manifold. Show that p is a fiber bundle.

Remark. It is not clear to me under what conditions p is a fiber bundle. A related question: can $P\mathbf{R}^n$ and $P\mathbf{R}^m$ be homeomorphic when n and m are different?

Problem 2. Let $\Omega X = \{\, \gamma \in X^I \mid \gamma(0) = \gamma(1) = x_0 \,\}$, and let $\gamma_0 \in \Omega X$ be the constant path. Show that $\pi_n(X, x_0) = \pi_{n-1}(\Omega X, \gamma_0)$. Give two different proofs: one direct, and one which uses the Serre fibration from the previous question. Define $\Omega(X, A) \subset PX$ in such a way that $\pi_{n-1}(\Omega(X, A), \gamma_0) = \pi_n(X, A, x_0)$.

Problem 3. Let $p : P \to B$ be fiber bundle. Let $f : B_1 \to B$ be a map. Then we can define a *pull-back* bundle

$$p_1 : p^*(P) \to B_1$$

by the recipe

$$p^*(P) = \{ (z, x) \in P \times B_1 \mid p(z) = f(x) \}$$

and $p_1(z, x) = x$. Show that p_1 really is a fiber bundle.

Now let $p : P \to S^2$ be a bundle with fiber S^1. Choose orientations for the fiber and the base. Then the long exact sequence of the bundle gives a map

$$\pi_2(S^2) \xrightarrow{\partial} \pi_1(S^1),$$

which (using the orientations) becomes a homomorphism $\Delta_p : \mathbf{Z} \to \mathbf{Z}$. Define the invariant $\delta(p) \in \mathbf{Z}$ to be $\Delta_p(1)$. This is the *degree* of the circle bundle over S^2. Show that the Hopf map $S^3 \to S^2$ is a circle bundle of degree 1, and use the pull-back construction to construct bundles of any given degree.

Problem 4. Let

$$P = \{ (u, v) \in S^n \times S^n \mid u \text{ and } v \text{ are orthogonal} \}.$$

Exhibit P as a bundle over S^n with fiber S^{n-1}.

Generalize the notion of degree from the previous question, to define the degree of an S^{n-1} bundle over S^n, and calculate the degree of P.

Show that P has the same homotopy groups as $S^n \times S^{n-1}$ if and only if n is odd.

(By "the same homotopy groups" I simply mean that the groups $\pi_i(P)$ and $\pi_i(S^n \times S^{n-1})$ are isomorphic for all i.)

Problem 5. The unitary group $U(n)$ acts on the unit sphere S^{2n-1} in $\mathbf{C}^n$. The stabilizer of $(0, \dots, 0, 1)$ is $U(n-1)$. In this way, we regard each $U(n)$ as contained in the next, and we write write $U = \bigcup U(n)$. The group $O = \bigcup O(n)$ and $Sp = \bigcup Sp(n)$ are constructed similarly. (The group $Sp(n)$ is the group of automorphisms of the $\mathbf{H}$-module $\mathbf{H}^n$ which also preserve the euclidean norm.) Bott's famous periodicity theorem states that

$$\pi_i(U) \cong \pi_{i+2}(U)$$
$$\pi_i(O) \cong \pi_{i+4}(Sp)$$
$$\pi_i(Sp) \cong \pi_{i+4}(O)$$

for $i \geq 0$. Deduce from the periodicity theorem the statement $\pi_4(S^3) = \mathbf{Z}/2$.

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